A new compound lead time demand distribution approach and a comparison study

by

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Abstract

The aim of this paper is to present an alternative lead time distribution called the mixed atom-delay distribution based on the idea of modelling lead time as a delay. Using this atom-delay distribution for lead time, a new compound lead time demand distribution approach is developed. The accuracy of reorder level determination in a continuous review inventory policy using this new approach is compared with a number of traditional decision rules.

Keywords
Inventory control, lead time, lead time delay, lead time demand, decision rules

1 Introduction

In most inventory and production planning problems there is an interval of time between the decision to place an order for more stock and the availability of the stock from that order to meet customer demand or a production setup. This time interval is called the lead time.

Traditionally, lead times are either assumed fixed or independent identically distributed (i.i.d.) random variables. In the first case a delay can not occur, since it is indirectly assumed that the supplier has infinite capacity and therefore never runs out of stock. If lead times are allowed to vary the common assumption is that lead times are i.i.d. A common procedure is then to estimate mean and standard deviation of the demand distribution (assuming stationary demand) and of the lead time distribution based on historical data. Then the parameters are combined to get the parameters of the compound distribution of demand during lead time. In standard textbooks on inventory theory (see e.g. Silver, Pyke and Peterson [32]) control policies are based on the assumption that the compound distribution of demand during lead time is (approximately) normal. In their paper, Bagchi, Hayya and Chu [2] provide an extensive discussion with examples showing that this assumption is seldom reasonable and can lead to high cost penalties and incorrect customer service levels. Therefore, they provide a review of literature that suggest models based on other lead time demand distributions.

An alternative approach to modelling lead time is to assume that the lead time normally takes a fixed value, but occasionally the order may be delayed. A lead time delay is therefore defined as the number of time units an order is delayed compared to the expected or promised lead time. One could imagine that delays usually are caused by a stockout situation at the supplier or a manufacturing breakdown. However, also defect goods, quality disputes, missing components or problems with transportation can cause additional delays. Hence, the alternative approach to modelling lead time is to assume that the lead time normally takes a constant value but the order may be delayed according to some probability distribution.

This paper is organised as follows. In Section 2 the idea of modelling lead time as a mixture of a fixed lead time and a delay is presented. This idea is called the mixed atom-delay distribution of lead time. Also in this section the compound lead time demand distribution approach is developed. In Section 3 the compound lead time demand distribution of normally distributed demand per period and exponentially distributed delays is developed. Based on this specific compound lead time demand distribution the decision rules for determining reorder levels for two service measures are developed. The two service measures are the cycle service level (i.e. the no-stockout probability) and the fill rate. The remainder of this paper will then focus on approaches to approximate the lead time demand distributions. The main goal is to compare the performance and accuracy of these different approaches. This will be done in Section 6. Prior to this comparison a number of approaches will be presented. In Section 4, decision rules based on lead time demand distributions will be presented, and in Section 5 other approximate decision rules will be presented and discussed. Finally, in Section 7 a summary and the concluding remarks are provided.

2 The mixed atom-delay distribution of lead time

As presented above an alternative approach to modelling lead time is to assume that the lead time normally takes a constant value but the order may be delayed according to some probability. Hence, the lead time,L, is defined as

$$L = \begin{cases} L_0 & \text{with probability } p_0 \text{ (fixed lead time)} \\ L_0 + t & \text{with probability } 1 - p_0 \text{ (variable lead time)} \end{cases}$$

The idea of modelling lead time as a delay is therefore based on some probability for a certain fixed lead time, which corresponds to the minimum possible lead time, and some probability that there may be a delay, so that the lead time gets longer than the minimum specified. In this kind of probability distribution, there will be an atom in $L = L_0$ specified by the mass p_0 and some probability distribution for $L > L_0$ with mass $1 - p_0$. This type of distribution will now be referred to as the mixed atom-delay lead time distribution. Modelling lead time like this can be illustrated as in Figure 1 below, where the delay is represented by two different probability distributions.

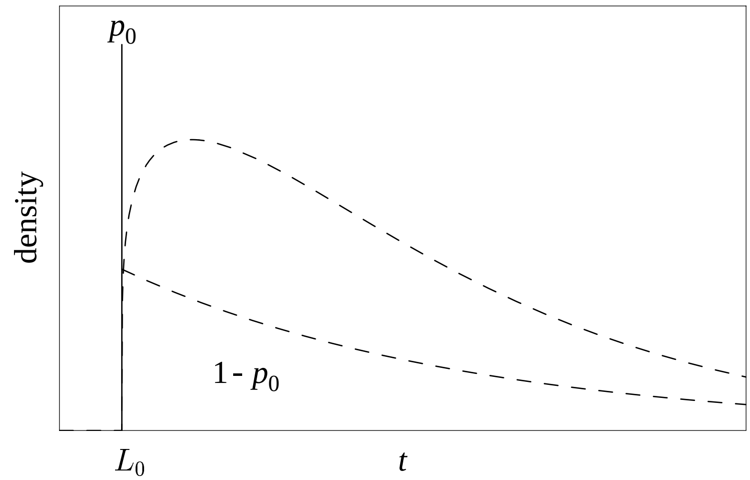

Figure 1: Lead time delay distribution

The traditional inventory policies, where lead time is either assumed fixed or assumed to follow some probability distribution, correspond to $p_0 = 1$ and $p_0 = 0$, respectively. Hence, for the fixed lead time there is no variation, and the whole mass ($p_0 = 1$) is concentrated at the fixed lead time, L_0. At the other end of the scale we have $p_0 = 0$, where the lead time is always considered as a kind of delay, and there is no lead time value having any atom. In this perspective the mixed atom-delay lead time distribution is a combination of having some probability of a fixed lead time and some probability of a variable lead time (the delay). As shown in Gudum [10], the realised lead time (as faced by end-user customers), which is estimated both from empirical data and from theoretical distribution systems, can in fact be modelled according to the mixed atom-delay distribution.

To derive the compound lead time demand distribution we need to define some relevant density functions:

$f(x)$	is the probability density function of demand per period.
$f(x\|L)$	is the probability density function of demand given L periods of lead time, which is the L-fold convolution of $f(x)$.
$g(t)$	is the probability density function of a delay.
$h(x)$	is the compound density function of the lead time demand given that there is a delay.
$k(x)$	is the mixed compound density function of the lead time demand.

Therefore,

$$h(x) = \int_0^\infty f(x|L_0 + t)g(t)dt$$

and

$$k(x) = p_0 f(x|L_0) + (1 - p_0)h(x)$$

From this mixed compound lead time demand distribution the no-stockout probability, $\alpha(s)$ and the backorder function, $B(s)$, can be derived, both as functions of the reorder level.

The no-stockout probability, defined as the probability of not having a stockout during the replenishment cycle (also called the cycle service level), is

$$\begin{aligned}\alpha(s) &= \int_0^s k(x)dx \\ &= p_0 \int_0^s f(x|L_0)dx + (1-p_0)\int_0^s \int_0^\infty f(x|L_0+t)g(t)dtdx\end{aligned} \tag{1}$$

The special case, where $L_0 = 0$, leads to

$$\alpha(s) = p_0 + (1-p_0)\int_0^s \int_0^\infty f(x|t)g(t)dtdx \tag{2}$$

because $f(x|0)$ is a so-called singular distribution. The lead time demand density $k(x)$ is then a mixed distribution in the sense that it has an atom of mass p_0 for zero lead time demand and the remainder $(1-p_0)$ of the mass follows a continuous distribution.

If $L_0 = 0$, the compound lead time demand distribution given a positive delay, $h(x)$, is equal to the traditional compound lead time demand distribution given in the literature (see for instance Burgin [4]), where $L = t$, since then

$$h(x) = \int_0^\infty f(x|t)g(t)dt$$

However, for $L_0 > 0$, the evaluation of $h(x)$ is more complicated due to the (L_0+t)-fold convolution of $f(x)$ combined with the integration only by t and not by $L_0 + t$.

The expected backorder per replenishment cycle is

$$\begin{aligned}B(s) &= \int_s^\infty (x-s)k(x)dx \\ &= p_0\int_s^\infty (x-s)f(x|L_0)dx + (1-p_0)\int_s^\infty \int_0^\infty (x-s)f(x|L_0+t)g(t)dtdx\end{aligned} \tag{3}$$

Technically, the reorder level could be negative. Therefore, for $L_0 = 0$, the first integral of the second line of Equation (3) has to be divided into two cases as to whether the reorder level is non-negative or negative. Hence, (for $L_0 = 0$)

$$B(s) = \begin{cases} (1-p_0)\int_s^\infty \int_0^\infty (x-s)f(x|L_0+t)g(t)dtdx + 0 & \text{if } s \geq 0 \\ (1-p_0)\int_s^\infty \int_0^\infty (x-s)f(x|L_0+t)g(t)dtdx - p_0 s & \text{if } s < 0 \end{cases} \tag{4}$$

In some literature the backorder function is denoted by the potential loss function. It also corresponds to the expected shortage per replenishment cycle, which in Silver et al. [32] is denoted by ESPRC.

From $B(s)$ the fill rate can be determined, also as a function of the reorder level. Let Q be the order size, then the fill rate is equal to

$$\beta(s) = 1 - \frac{1}{Q}B(s) \tag{5}$$

The input to the service functions, $\alpha(s)$ and $\beta(s)$, are the distribution parameters of $f(x)$ and $g(t)$, the fixed (minimum) lead time, L_0, the probability of a fixed lead time, p_0, the reorder level, and finally for the fill rate also the order size. Some types of compound distributions can be inverted to be functions of a target service level instead of the reorder level. The inverse of Equations (1) and (5) are here called reorder functions of target α and β, respectively. The service functions and, if available, the reorder functions comprise the decision rules for determining the reorder level subject to a service level constraint.

3 Decision rules using the mixed atom-exponential-delay distribution

In this section, the service functions will be derived for the case of normally distributed demand and exponentially distributed delays. Each of the service functions will be derived separately for the cases where $L_0 = 0$ and $L_0 > 0$. Let μ_D and σ_D denote the mean and standard deviation of demand per period, and let λ denote the mean exponential delay, where λ is the parameter of the exponential delay distribution. Then,

$$g(t) = \frac{1}{\lambda} e^{\frac{-t}{\lambda}}$$

and

$$f(x|L_0 + t) = \frac{1}{\sigma_D \sqrt{2\pi(L_0 + t)}} e^{-\frac{1}{2}\left(\frac{x-(L_0+t)\mu_D}{\sigma_D\sqrt{L_0+t}}\right)^2}$$

Decision rules for $L_0 = 0$

For $L_0 = 0$, the lead time demand distribution given a delay, $h(x)$, corresponds to the classical situation of the compound lead time demand distribution presented in Burgin [4]. Therefore, $h(x)$ is determined as

$$h(x) = \frac{1}{\lambda\theta} e^{-x\omega}$$

and

$$\int_0^s h(x)dx = 1 - \frac{1}{\lambda\theta\omega} e^{-x\omega}$$

where $\theta = \sqrt{\mu_D^2 + (2\sigma_D^2)/\lambda}$, and $\omega = \frac{\theta - \mu_D}{\sigma_D^2}$.

Based on Equation (2) the P_1 service function (cycle service level) given a specified reorder level becomes

$$\alpha(s) = p_0 + (1 - p_0)\left(1 - \frac{1}{\lambda\theta\omega} e^{-s\omega}\right) \tag{6}$$

It can now easily be shown that the reorder level given a target cycle service level, α, is

$$s(\alpha) = \frac{1}{\omega} \ln\left(\frac{1 - p_0}{(1 - \alpha)\lambda\theta\omega}\right) \tag{7}$$

If $p_0 = 0$ this equation corresponds to the classical case of exponentially distributed lead times.

The P_2 service function (fill rate) given a specified reorder level is found by Equation (5), where $B(s)$ using Equation (4) becomes

$$B(s) = \begin{cases} (1-p_0)\frac{1}{\sigma_D \lambda\sqrt{2\pi}} \int_s^\infty \int_0^\infty (x-s)\frac{1}{\sqrt{t}} e^{-\frac{t}{\lambda}-\frac{1}{2}\left(\frac{x-t\mu_D}{\sigma_D\sqrt{t}}\right)^2} dt dx & \text{if } s \geq 0 \\ (1-p_0)\frac{1}{\sigma_D \lambda\sqrt{2\pi}} \int_s^\infty \int_0^\infty (x-s)\frac{1}{\sqrt{t}} e^{-\frac{t}{\lambda}-\frac{1}{2}\left(\frac{x-t\mu_D}{\sigma_D\sqrt{t}}\right)^2} dt dx - p_0 s & \text{if } s < 0 \end{cases} \tag{8}$$

Decision rules for $L_0 > 0$

Integration of the lead time demand density function is more complicated for $L_0 > 0$, and there is no closed-form representation. However, with advanced programs such as *Mathematica*®, the service functions can be solved for numerical data.

Based on Equation (1) the P_1 service function (cycle service level) becomes

$$\begin{aligned} \alpha(s) = {} & p_0 \frac{1}{\sigma_D\sqrt{2\pi L_0}} \int_0^s e^{-\frac{1}{2}\left(\frac{x-L_0\mu_D}{\sigma_D\sqrt{L_0}}\right)^2} dx \\ & + (1-p_0)\frac{1}{\sigma_D\lambda\sqrt{2\pi}} \int_0^s \int_0^\infty \frac{1}{\sqrt{L_0+t}} e^{-\frac{t}{\lambda}-\frac{1}{2}\left(\frac{x-(L_0+t)\mu_D}{\sigma_D\sqrt{L_0+t}}\right)^2} dt dx \end{aligned} \tag{9}$$

and based on Equation (3) the backorder function becomes

$$\begin{aligned} B(s) = {} & p_0 \frac{1}{\sigma_D\sqrt{2\pi L_0}} \int_s^\infty (x-s) e^{-\frac{1}{2}\left(\frac{x-L_0\mu_D}{\sigma_D\sqrt{L_0}}\right)^2} dx \\ & + (1-p_0)\frac{1}{\sigma_D\lambda\sqrt{2\pi}} \int_s^\infty \int_0^\infty (x-s)\frac{1}{\sqrt{L_0+t}} e^{-\frac{t}{\lambda}-\frac{1}{2}\left(\frac{x-(L_0+t)\mu_D}{\sigma_D\sqrt{L_0+t}}\right)^2} dt dx \end{aligned} \tag{10}$$

Finally, the P_2 service function (fill rate) is determined by Equation (5).

Since these equations can be evaluated for numerical data in *Mathematica*®, it is possible to find a computational procedure such that the approach could be implemented in inventory management systems. The goal of this paper, however, is not to represent implementable computational procedures, but rather to compare the performance of this approach with traditional methods. If the applicability of this approach is justified, future research is suggested into how this procedure can be implemented in practice.

4 Decision rules based on lead time demand distributions

There are a number of approaches to modelling the lead time demand. Tyworth and O'Neil [38] divide the research of modelling lead time demand into four lines of thoughts. The first centers on determining the compound probability distribution analytically based on theoretical distributions of demand and lead time. This approach has been reviewed by Bagchi, Hayya and Ord [3] and by Bagchi, Hayya and Chu [2]. Also, the atom-delay distribution approach proposed in the previous sections belongs to this category. The second line of thought, which will be explored in this section, focuses on the use of various statistical forms to model the lead time demand distribution directly. The third line of thought, which will be in focus in the next section, is based on distribution free approaches, where inventory problems are solved without imposing restrictive assumptions about the lead time demand distribution. These three lines of thought all assume that the lead time demand is important for the optimal inventory decision making. Within the fourth line of thought it is argued that the shape of the lead time demand distribution is not important. For instance, Naddor [25] indicates that the mean and standard deviation of demand is much more important to optimal decision making than the shape of the probability distribution of lead time demand. Hence, the use of the normal distribution is argued to be adequate. Counter examples, however, have been made by Lau and Zaki [20] and Heuts et al. [12], among others, showing that the wrong density function indeed causes serious errors in estimating the optimal inventory decisions thereby advocating for the use of one of the first three lines of thoughts.

This section is focused on the second line of thought, and several distributions have been suggested for approximating the lead time demand distribution in inventory systems directly. The normal distribution is recommended in most inventory related textbooks (e.g. [32]). In simultaneous optimisation of the reorder point and the order size, van Beek [39] uses the logistic distribution function as an alternative to the normal because of the simpler mathematical expressions it yields. A number of studies focus on asymmetric distributions. For instance, Burgin [5] uses the gamma distribution to model lead time demand, and Tadikamalla [34] shows that the Weibull distribution can be used to approximate the lead time demand distribution emphasizing the simple computations involved. Kottas and Lau [16, 17] focus on distributions based on the first four central moments of the lead time demand and therefore propose the Schmeiser-Deutch family of curves and the beta distribution to fit the lead time demand. Lau [19] proposes the Pearson family of distributions to fit the lead time demand, and Kumaran and Achary [18] suggest the generalised lambda type of distributions based on the work of Ramberg et al. [27]. Hence, much focus is beyond the first two moments of the lead time demand distribution.

The first four moments are the most observable characteristics of a probability distribution. These are the location (mean), the dispersion (variance), the skewness and the kurtosis. As pointed out in Johnson, Kotz and Balakrishnan [13], Pearson classified distributions into types of solutions to differential equations. The idea is that the combination of skewness (β_1) and kurtosis (β_2) defines the type of distribution. Kottas and Lau [17] represent a (β_1, β_2)-diagram to highlight the versatility of the Schmeiser-Deutch, the Pearson and the beta distributions opposed to the normal and exponential distributions that only occupy single points, while the gamma and the Weibull are represented by lines.

Fortuin [8] compares five distributions, namely the normal, the logistic, the gamma, the lognormal and the Weibull distributions. From this comparison study it is concluded that in fact the difference between inventory decisions of these distributions is very small. Therefore, Fortuin recommends the use of the logistic distribution due to its advantage of a closed-form solution, which implies that it is easy to implement in inventory management systems. Tadikamalla [35], however, also compares these five distributions and show that the coefficient of variation of lead time demand is an important factor for optimal inventory decisions. Tadikamalla then concludes that the normal and the logistic distribution are inadequate to represent the lead time demand, since the inventory decisions are sensitive to the shape of the distribution.

As has now been pointed out, much attention is paid toward the shape of the lead time demand density whether it is based on a two-parameter distribution, such as the gamma and the Weibull, or a four-parameter distribution, such as the beta, Pearson and Schmeiser-Deutch distributions.

If observed data of the lead time demand, x, is available, the first four central moments, determined from the sample data, are

$$\begin{aligned}
\mu &= E(x) \\
\mu_2 &= \sigma^2 = E(x-\mu)^2 = E(x^2) - \mu^2 \\
\mu_3 &= E(x-\mu)^3 = E(x^3) + 2\mu^3 - 3E(x^2)\mu \\
\mu_4 &= E(x-\mu)^4 = E(x^4) - 3\mu^4 - 4E(x^3)\mu + 6E(x^2)\mu^2
\end{aligned}$$

If observed values of the lead time demand are not available, but observed data for the period demand and lead time are available, then the first four central moments of the lead time demand can be estimated from the first four central moments of period demand and lead time. Let μ_D, σ_D^2, $\mu_3(D)$ and $\mu_4(D)$ denote the first four central moments of demand per period and in the same way, let μ_L, σ_L^2, $\mu_3(L)$ and $\mu_4(L)$ denote the first four central moments of the lead time. Then (see Lau and Zhao [21]),

$$\begin{aligned}
\mu &= \mu_D\mu_L \\
\mu_2 &= \sigma^2 = \mu_D^2\sigma_L^2 + \mu_L\sigma_D^2 \\
\mu_3 &= \mu_D^3\mu_3(L) + \mu_3(D)\mu_L + 3\mu_D\sigma_D^2\sigma_L^2 \\
\mu_4 &= \mu_D^4\mu_4(L) + 6\mu_D^2\sigma_D^2(\mu_L\sigma_L^2 + \mu_3(L)) \\
&\quad + 4\mu_D\mu_3(D)\sigma_L^2 + \mu_4(D)\mu_L + 3\sigma_D^4(\mu_L^2 - \mu_L + \sigma_L^2)
\end{aligned}$$

Based on the first four central moments, the skewness and kurtosis can be estimated:

$$\alpha_3 = \frac{\mu_3}{\mu_2^{1.5}}$$
$$\alpha_4 = \frac{\mu_4}{\mu_2^2}$$

where $\beta_1 = \alpha_3^2$ and $\beta_2 = \alpha_4$ also represent measures of skewness and kurtosis.

In Table 1 below, a number of decision rules based on lead time demand distributions are presented, which will be used for the comparison study in Section 6. Each decision rule is based on the determination of the reorder level given a target service level, where α and β denote the target service levels of the P_1 and P_2 service measure, respectively. For some distributions or for some service types, however, it is only possible to derive a closed-form of the service function or the backorder function. For those cases a trial-and-error search to find the optimal reorder point for a given target service level is necessary. In the cases where $s(\beta)$ is not defined, $B(s)$ will be presented instead, where $\beta(s) = 1 - B(s)/Q$. Moreover, 'LTD' is an abbreviation for lead time demand. The generalised lambda distribution and the Pearson approach are only developed for the P_1 service measure.

To determine the reorder level based on these lead time demand distributions, the parameters of each distribution must be estimated from observed data. Table 2 below summarises the estimation procedure for each of the chosen distributions using method of moments. An alternative way is to use maximum likelihood estimation, which, however, requires software including these kinds of estimation procedures.

LTD Distribution	Decision Rules
Normal $N(\mu, \sigma)$	P_1: $s = \mu + \Phi^{-1}(\alpha)\sigma$ P_2: $s = \mu + k\sigma$ where k satisfies $\phi(k) - k + k\Phi(k) = (1-\beta)Q/\sigma$
Gamma $\Gamma(\lambda_1, \lambda_2)$	These formulas are based on results of Burgin [5] P_1: For $2\lambda_1 \geq 1$: $s = \lambda_2 \chi^2_\alpha(2\lambda_1)/2$ Otherwise: $\alpha(s) = I_s(s/(\lambda_2\sqrt{\lambda_1}), \lambda_1 - 1)$ P_2: $B(s) = (\lambda_1\lambda_2 - s)(1 - I_s(s/(\lambda_2\sqrt{\lambda_1}), \lambda_1 - 1))$, $s \geq 0$ $+\lambda_1\lambda_2(s/\lambda_2)^{\lambda_1} e^{-s/\lambda_2}/\Gamma(\lambda_1 + 1)$ where I_s denotes the incomplete gamma function ratio of s.
Exponential $\text{Exp}(\lambda)$	P_1: $s = \lambda \ln(1/(1-\alpha))$ P_2: $s = \lambda \ln(\lambda/(Q(1-\beta)))$
Weibull $\text{Weib}(b, c)$	These formulas are derived in Tadikamalla [34] P_1: $s = b(\ln(1/(1-\alpha)))^{1/c}$ P_2: $B(s) = b\Gamma(q)(1 - I_s(p, q)) - se^{-(s/b)^c}$ where $p = \sqrt{c/(1+c)}(s/b)^c$ and $q = (1+c)/c$ and I_s denotes the incomplete Gamma function ratio of s.
Beta $\text{Beta}(a, b, p, q)$	These formulas are derived in Kottas and Lau [17] P_1: $\alpha(s) = I_r(p, q)$ P_2: $B(s) = (1 - I_r(p+1, q))(b-a)\frac{p}{p+q} - (s-a)(1 - I_r(p, q))$ where $r = (s-a)/(b-a)$ and $s \geq a \geq 0$ and I_r is the incomplete Beta function ratio of r.
Schmeiser-Deutch $\text{SD}(a, b, c, d)$	These formulas are derived in Kottas and Lau [17] P_1: $s = a - b(d-\alpha)^c$ if $\alpha \leq d$; $s = a + b(\alpha - d)^c$ if $\alpha > d$ P_2: if $s \geq a$: $B(s) = \frac{b}{c+1}\big((1-d)c + 1 - (\frac{s-a}{b})^q\big) - (s-a)\big(1 - d - (\frac{s-a}{b})^m\big)$ if $s < a$: $B(s) = \frac{b}{c+1}\big((1-d)c + 1 + (\frac{a-s}{b})^q\big) + (a-s)\big(1 - d + (\frac{a-s}{b})^m\big)$ where $q = (1+c)/c$ and $m = 1/c$
Pearson $(\mu, \sigma, \alpha_3, \alpha_4)$	This formula is presented in Kottas and Lau [16] P_1: $s = \mu + k\sigma$, where k is determined from JNAP-tables in [14]
Generalised λ $x \sim \lambda(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$	This formula is derived in Kumaran and Achary [18] P_1 : $s = \lambda_1 + \frac{\alpha^{\lambda_3} - (1-\alpha)^{\lambda_4}}{\lambda_2}$

Table 1: Decision rules based on various lead time demand distributions

LTD Distribution	Parameter estimation by method of moments
Normal	μ and σ determined either from sample data of LTD or compounded from sample data of D and L
Gamma	$\lambda_1 = \mu^2/\sigma^2$ (shape parameter) $\lambda_2 = \sigma^2/\mu$ (scale parameter)
Exponential	$\lambda = \mu$
Weibull	$c = f^{-1}(\mu/\sigma)$ (shape parameter) where $\frac{\mu}{\sigma} = f(c) = \frac{\Gamma(1+1/c)}{\sqrt{\Gamma(1+2/c) - \Gamma(1+1/c)^2}}$ is easily tabulated. $b = \frac{\mu}{\Gamma(1+1/c)}$ (scale parameter)
Beta	$a =$ minimum observed LTD value $b =$ maximum observed LTD value $p = \frac{\mu_s^2(1-\mu_s)}{\sigma_s^2} - \mu_s$ $q = \frac{p(1-\mu_s)}{\mu_s}$ where $\mu_s = \frac{\mu - a}{b-a}$ and $\sigma_s = \frac{\sigma}{b-a}$
Schmeiser-Deutch	$\theta_1 = \frac{\alpha_3^2}{(1+\alpha_3^2)}$ $\theta_2 = \frac{1}{\alpha_4}$ Based on θ_1 and θ_2: c and d are read off from a figure, which is represented in Kottas and Lau [16] $b = \sqrt{\frac{\sigma^2 p^2 q}{p^2(d^q + (1-d)^q) - q((1-d)^p - d^p)^2}}$ $a = \mu - \frac{b((1-d^p - d^p)}{p}$ where $p = c+1$ and $q = 2c+1$
Pearson	is based directly on $(\mu, \sigma, \alpha_3, \alpha_4)$.
Generalised λ	The estimation of $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ is based on tables, where α_3 and α_4 are input. Such tables are given in Ramberg et al. [26, 27].

Table 2: Estimation of parameters in the lead time demand distributions

5 Decision rules based on distribution free approaches

As was pointed out in the previous section, the lead time demand can be modelled in a number of ways. In this section we will focus on the distribution free approaches, where inventory problems are solved without imposing restrictive assumptions about the lead time demand distribution.

A clear disadvantage of using the compound distribution approach is that there exists no universal compound distribution, which is applicable for a wide range of situations. As seen in Gudum [10] each combination of demand and lead time distribution leads to a different compound distribution.

Eppen and Martin [7], moreover, question the assumption that the distributions of demand and lead time are known. Based on estimations of the first two moments, they focus on how to set demand parameters in the forecasting model with stochastic lead time and how to determine the safety stock from these parameters.

Based on Eppen and Martin's perspective, Tyworth [36] proposes a new paradigm for inventory modelling with variable lead times. He comments on a number of limitations of the compound distribution approach, which he calls the "current paradigm". The most important limitation is that modelling the true form of the lead time demand as a compound distribution is indeed very difficult. The procedure proposed by Tyworth's new paradigm, is a convex combination of conditional demand distributions. First, a number of possible lead time values are identified. The demand distribution can either be normal or Poisson. Then the L-fold convolution of the demand distribution is computed for each level of the lead time, L. This leads to the lead time demand distribution for each possible lead time level. Then the expected shortages per replenishment cycle for each lead time level is estimated and weighted by the probability of each lead time value. The weighted average is then used to determine the optimal reorder point for a given fill rate or shortage cost. The advantage of this new approach is that the safety stock can be determined without any knowledge of the shape of L. This convex combination approach will be used in the comparison study, and below an adaptation to the delay situation is presented. Keaton [15] extends this paradigm for the gamma distributed demand per period, and Tyworth et al. [37] then provide a spreadsheet based approach for finding the compound distribution of demand during lead time.

An alternative procedure based on order statistics is to measure the total demand over the lead time. The procedure is developed by Lordahl and Bookbinder [22] and is very simple. The objective is to determine the reorder point, s, when the parameters and the shape of the lead time demand distributions are unknown. The procedure determines s as the α'th quantile in the empirical distribution of lead time demand. The advantage of this procedure is therefore that it is based on distribution-free properties of order statistics. The disadvantage is that this procedure is only developed for the cycle service measure. The procedure, however, can easily be extended to the fill rate service measure as we will show in Table 3 below.

Another approach is the minmax procedure, originally applied by Scarf [28] for the newsboy problem and later used for the (s, Q)-type of policies by Moon and Choi [23] and Moon and Gallego [24]. The minmax procedure is a conservative approach that optimises the inventory decision variables against the worst possible distribution of lead time demand based only on the mean and variance. Shore [29, 30, 31] also provides some approximations to the reorder level when information about the lead time demand distribution is limited. Moreover, Strijbosch and Heuts [33] suggest the use of a non-parametric kernel density approach, which is very close to the shape of a histogram of scarce empirical data.

In Table 3 below, decision rules based on some of the distribution free approaches are presented. These decision rules will be part of the comparison study.

Approach	Decision Rule
Order statistics	This approach is proposed by Lordahl and Bookbinder [22]: P_1: n observed LTD's are sorted in increasing order so that $x_{(1)} \leq ... \leq x_{(n)}$. $r + w \equiv (n+1)\alpha$ where $0 \leq w < 1$ and r is a non-negative integer $s = (1-w)x_{(r)} + wx_{(r+1)}$ if $0 < w < 1$ $s = x_{(r)}$ if $w = 0$ $s = x_{(n)}$ if $(n+1)\alpha > n$ We propose an extension of the above: P_2: n observed LTD's are sorted in increasing order so that $x_{(1)} \leq ... \leq x_{(n)}$. An initial value of the reorder point, s, is chosen. For each $i = 1, ..., n$, $B(s, x_i) = \max(0, s - x_i)$ is evaluated. Then $B(s) = \frac{1}{n}\sum_{i=1}^{n} B(s, x_i)$ The fill rate is then $\beta = 1 - B(s)/Q$ s is chosen so that the target fill rate is obtained. This approach is easily implemented into a spreadsheet. (The observations need not necessarily be sorted)
Convex combination	This approach is proposed by Tyworth [36]: P_1: $\alpha = \sum_{i=0}^{K} \Phi(k_i)p_i$ P_2: $\beta = 1 - \frac{1}{Q}\sum_{i=0}^{K} G(k_i)\sigma_i p_i$ Notation and an adaptation of this approach to the delay distribution is presented below.
Moon&Choi [23]	This approach is only developed for the fill rate P_2: $s = \mu + \frac{\sigma^2}{4(1-\beta)Q} - (1-\beta)Q$
Fixed lead times	P_1: $s = \mu + \Phi^{-1}(\alpha)\sigma$ P_2: $s = \mu + k\sigma$ where k satisfies $\phi(k) - k + k\Phi(k) = (1-\beta)Q/\sigma$ and $\mu = \mu_L\mu_D$ and $\sigma = \sqrt{\mu_L}\sigma_D$

Table 3: Decision rules based on distribution free approaches

Adapting Tyworth's paradigm to the delay perspective

The lead time was defined as $L = L_0 + t$, where t equals 0 with probability p_0, and $t > 0$ with probability $1 - p_0$. From the empirical lead time data we can calculate the fraction of customers who did not experience any delay: $p_0 = P(t = 0)$, i.e. the probability of no delay. From the delay distribution analysis we have both empirical and fitted probabilities of each possible delay value. For the empirical lead time data, the empirical delay distribution is discrete with the delay taking integer values. Hence, the delay can take values $t_1, ..., t_K$, where t_K is the maximum observed value and t_1 is the minimum (positive) delay. Let $t_0 = 0$ denote the situation of no delay. Given a delay, the probability of a delay of t_i is $g(t_i)$ found by the distribution analysis, where the sum of $g(t_i)$'s equals unity. Then the probability of a certain lead time, $L_i = L_0 + t_i$, is $p_i = (1 - p_0)g(t_i)$ for $i > 0$ and p_0 for $i = 0$.

This information is used to generate a convex combination of conditional period demand distributions. Period demand is assumed normally distributed with mean μ_D and standard deviation σ_D. Calculate the conditional mean and standard deviation for each lead time value (i=0,..., K) as

$$\mu_i = L_i \mu_D$$
$$\sigma_i = \sqrt{L_i}\sigma_D$$

corresponding to an L_i-fold convolution of the demand distribution for each possible value of the lead time. Now, given the value of the reorder point, the safety factor is determined for each of the lead time values as

$$k_i(s) = \frac{s - \mu_i}{\sigma_i}$$

For each $k_i(s)$ the partial expectation function, $G_u(k_i(s))$, can be determined by $G_u(k_i(s)) = \phi(k_i(s)) - k_i(s)(1 - \Phi(k_i(s)))$. Here $\phi(k_i(s))$ is the standard normal density of $k_i(s)$, and $\Phi(k_i(s))$ is the standard normal cumulative distribution function of $k_i(s)$. The backorder function, which is the expected shortage per replenishment cycle, can then be found for each i by multiplying $G_u(k_i(s))$ by σ_i. This value is then weighted by the probability of each lead time value. Let $B(s)$ denote the overall expected shortage per replenishment cycle, then

$$B(s) = \sum_{i=0}^{K} G_u(k_i(s))\sigma_i p_i$$

Let α denote the target cycle service level and β the target fill rate. The decision rules for the cycle service level and the fill rate, respectively, are:

$$\text{choose } s \text{ so that } \alpha = \sum_{i=0}^{K} \Phi(k_i(s))p_i$$

$$\text{or choose } s \text{ so that } \beta = 1 - \frac{\sum_{i=0}^{K} G_u(k_i(s))\sigma_i p_i}{Q},$$

where Q is a predetermined order size. A search routine based on non-linear solution approach or trial-and-error can be applied to find the value of s given some target service level.

The convex combination of the conditional period demand distributions is illustrated in Figure 2 below, where each curve corresponds to a certain lead time value.

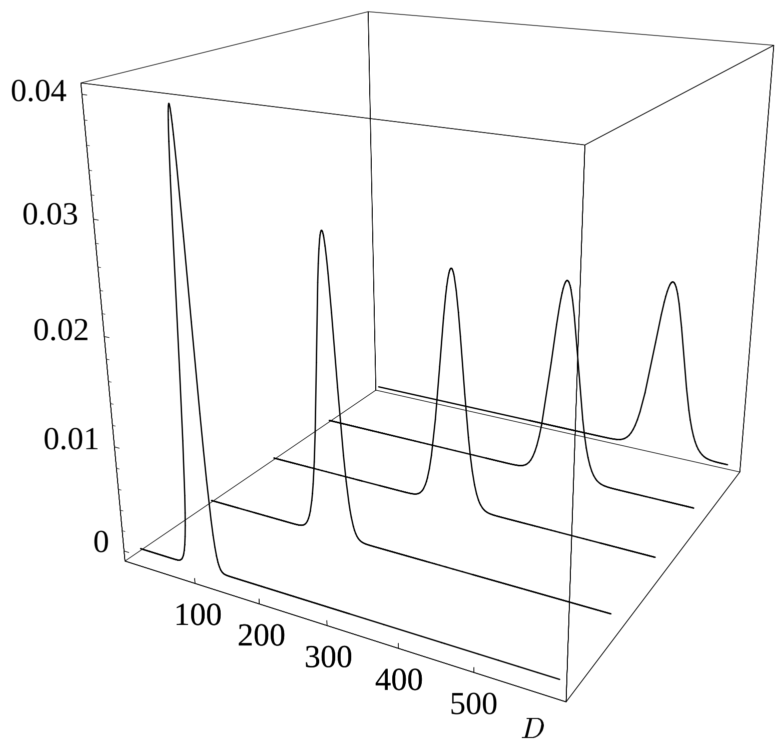

Figure 2: Convex combination of conditional demand distributions

Since Tyworth's procedure uses information about the empirical distribution of lead time, the associated reorder level is exact in the long run, here assuming normal demand.

6 A comparison study of lead time demand approximations

In this section decision rules based on different lead time demand approximations for determining the reorder level subject to a service level constraint will be analysed. A cost model will not be defined, and therefore no joint determination of the reorder point and the order quantity is attempted. Thus, we will not be concerned with the determination of Q. Instead we restrict our attention to the problem of determining the reorder point, s, in order to obtain a target service level.

In the previous sections, we have been concerned with different techniques for determining the reorder point based on assumptions of the lead time demand structure. The techniques are: (1) The atom-delay lead time distribution (with exponential delay), (2) Tyworth's convex combination approach, (3) Order statistics approach, (4) Normal LTD, (5) Gamma LTD, (6) Weibull LTD, (7) Exponential LTD, (8) Beta LTD, (9) Schmeiser-Deutch LTD, (10) Pearson LTD, (11) Generalised lambda LTD, (12) LTD with fixed lead times and (13) Moon&Choi (only for P_2). In this section we will test these techniques based on empirical lead time data.

6.1 Experimental design

Through simulation a sample path of lead time demands are generated based on normally distributed periodic demand and empirical lead time data. Hence, a simulation study is designed, that simulates the demand per period assuming a normal distribution with mean $\mu_D = 100$. The standard deviation, σ_D, is varied as an experimental factor to evaluate the techniques for different values of the coefficient of variation of demand. Three values of the standard deviation of demand is chosen: $\sigma_D = \{30, 50, 80\}$. Three empirical lead time samples are chosen to be analysed. The chosen lead time samples are picked from a case study on lead time delay distributions carried out in Gudum [10] so that various values of p_0 are represented. The three samples will be denoted A, B and C. For all three empirical distributions the minimum lead time is 0; hence $L_0 = 0$.

Sample C is used in two versions (v1 and v2). The first version is the sample data as they are, leading to a p_0-value of 0.74. To obtain a distribution that has a higher value of p_0, a number of observations with "0"-lead time has been added artificially in the second version of sample C, leading to $p_0 = 0.90$; however, the delay distribution is the same as for sample C(v1). The reason for this is to have data samples that give a wider spread of p_0-values. Now, these 4 empirical distributions each correspond to a different level of p_0, i.e. the probability that the replenishment lead time was fixed at $L_0 = 0$. To see the effect of $L_0 > 0$ on the reorder level, sample C was also applied in a third version, C(v3). The sample data of lead times from sample C(v2) were all added with 4 periods, to obtain $L_0 = 4$, still having $p_0 = 0.90$. The five final sets of empirical lead time data are summarised in Table 4 below.

Sample	p_0	L_0
A	0.44	0
B	0.69	0
C(v1)	0.74	0
C(v2)	0.90	0
C(v3)	0.90	4

Table 4: Empirical lead time data

The simulation is designed to simulate an inventory control process during 5,000 periods using an (R, s, Q) ordering policy, where R is equal to one period. Prior to each simulation an arbitrary initial reorder level is determined and both the net stock and the inventory position is initially set equal to the reorder level. In each period the actual demand is generated from the normal distribution truncated at 0. Hence, if the generated demand is negative it is rejected and a new value is generated. In each period it is evaluated whether the inventory position is at or below the reorder level. If so, an order of quantity $Q = 1,000$ units is placed. This order is associated with a lead time drawn randomly from the empirical lead time data set (adjusted against order crossing). During the lead time the actual demand generated is accumulated to give a lead time demand sample path.

Through a retrospective analysis, the reorder level is chosen such that a target service level is met. Therefore, each simulation is run three times. The first run determines the minimum and maximum net stock, the second run determines the empirical distribution of the net stock process, and based on this distribution the reorder level is adjusted to meet the target service level. Finally, the third simulation is run to verify the service level and to produce the lead time demand sample path. This kind of adjustment is possible, when the initial net stock is equal to the reorder level, since then the net stock process is independent of the reorder level. This methodology is further elaborated on in Gudum [11], where it is formally developed to adjust safety stocks for more general inventory systems. These three simulation steps are replicated 10 times with different seeds to get a representative reorder point. The purpose of this retrospective analysis is to find the exact reorder level that for a given lead time demand sample path produces target service levels. This exact reorder level then serves as a benchmark for comparison of the traditional techniques and the atom-delay distribution approach.

For the generated processes of periodic demand, lead time and lead time demand, the first four central moments are calculated for use in the various techniques in the comparison study. Also, based on the random lead times drawn from the empirical lead time sample, a discrete empirical lead time distribution is determined for use as input to Tyworth's [36] convex combination approach.

The experiment thus comprises three values of the standard deviation of demand and 5 sets of empirical lead time data. For each of these 15 combinations, the 12 techniques and the simulated reorder point benchmark are evaluated for $P_1 = 0.95$ and $P_2 = 0.98$. For P_2 actually 13 techniques and the benchmark are evaluated, since the Moon&Choi technique is also included.

6.2 Numerical results

During the computations it turns out that the values needed to estimate the parameters of the generalised lambda distribution and the Pearson distribution fall beyond those tabulated in the literature. Therefore, these two techniques are withdrawn from the comparison study. Moreover, the reorder point of the Schmeiser-Deutch approach is very sensitive to the reading of the value of the c-parameter. Since it is very difficult to determine c accurately from the figure in Kottas and Lau [16], the results of Schmeiser-Deutch are subject to big sensitivity. For the P_2 service measure, the Schmeiser-Deutch approach produces very odd results, which will not be reported due to the high sensitivity. The results of the Schmeiser-Deutch approach will be presented for the P_1 measure; however, no conclusive comments will be provided for this approach.

The mean and standard deviation of the generated processes of demand, lead time and lead time demand of the 15 scenarios are provided in Table 9 in Appendix A. To show

how the computations of the various decision rules are carried out, the computations for the P_1 service measure with $\sigma_D = 30$ and $p_0 = 0.44$ (empirical lead times of sample A) are presented in detail in Appendix B.

The computed reorder levels of the decision rules and the simulated benchmark are shown below in Tables 5 and 7 for the P_1 and the P_2 service measures, respectively. For the P_2 measure, the gamma approach is only evaluated for the case of $L_0 = 4$, since it is numerically intractable for the other cases due to the presence of irregular parts.

Sample	A		B		C(v1)		C(v2)		C(v3)	
L_0	0		0		0		0		4	
p_0	0.44	%	0.69	%	0.74	%	0.9	%	0.9	%
	s	error	s	error	s	error	s	error	s	error
$\sigma_D = 30$										
Benchmark	1125		449		698		160		593	
Atom-delay	1017	-9.6	461	2.7	655	-6.2	106	-33.8	541	-8.8
Tyworth	1194	6.1	448	-0.2	567	-18.8	105	-34.4	532	-10.3
Order statistics	1185	5.3	428	-4.7	557	-20.2	106	-33.8	529	-10.8
Normal LTD	1154	2.6	378	-15.8	654	-6.3	140	-12.5	567	-4.4
Gamma LTD	1247	10.8	400	-10.9	581	-16.8	76	-52.5	578	-2.5
Weibull LTD	1067	-5.2	337	-24.9	464	-33.5	67	-58.1	558	-5.9
Exponential LTD	702	-37.6	220	-51.0	298	-57.3	46	-71.3	1250	110.8
Beta LTD	1497	33.1	479	6.7	645	-7.6	69	-56.9	590	-0.5
Fixed LT	398	-64.6	114	-74.6	144	-79.4	31	-80.6	511	-13.8
SD LTD	1584	40.8	520	15.8	366	-47.6	49	-69.4	432	-27.2
$\sigma_D = 50$										
Benchmark	1138		575		859		211		700	
Atom-delay	1038	-8.8	545	-5.2	718	-16.4	222	5.0	673	-4.0
Tyworth	1157	1.7	531	-7.7	494	-42.5	128	-37.4	623	-11.4
Order statistics	1172	3.0	556	-3.3	656	-23.6	115	-43.5	612	-13.1
Normal LTD	1173	3.1	450	-21.7	684	-20.4	343	59.6	792	13.7
Gamma LTD	1261	10.8	480	-16.5	667	-22.4	101	-49.6	840	20.8
Weibull LTD	1074	-5.6	407	-29.2	542	-36.9	120	-40.8	823	18.3
Exponential LTD	705	-38.0	266	-53.7	348	-59.5	92	-53.8	1331	93.8
Beta LTD	1516	33.2	584	1.6	801	-6.8	56	-69.8	867	24.8
Fixed LT	341	-70.0	164	-71.5	194	-77.4	78	-59.9	610	-13.4
SD LTD	1535	34.9	589	2.4	171	-80.1	116	-42.8	539	-23.9
$\sigma_D = 80$										
Benchmark	1185		611		974		222		801	
Atom-delay	1264	6.7	556	-9.0	791	-18.8	128	-42.3	733	-8.5
Tyworth	1601	35.1	570	-6.7	760	-22.0	128	-42.3	721	-10.0
Order statistics	1536	29.6	569	-6.9	742	-23.8	117	-47.3	724	-9.6
Normal LTD	1369	15.5	454	-25.7	752	-22.8	192	-13.5	766	-4.4
Gamma LTD	1471	24.1	480	-21.4	678	-30.4	89	-59.9	796	-0.6
Weibull LTD	1254	5.8	405	-33.7	543	-44.3	84	-62.2	772	-3.6
Exponential LTD	823	-30.5	263	-57.0	349	-64.2	59	-73.4	1452	81.3
Beta LTD	1753	47.9	581	-4.9	750	-23.0	70	-68.5	793	-1.0
Fixed LT	443	-62.6	180	-70.5	224	-77.0	61	-72.5	703	-12.2
SD LTD	1880	58.6	595	-2.6	422	-56.7	67	-69.8	718	-10.4

Table 5: Reorder levels and percentage errors of decision rules for $P_1 = 0.95$

Overall and average absolute errors and ranks of approaches for the P_1 measure can be seen in Table 6 below.

	$\sigma_D = 30$		$\sigma_D = 50$		$\sigma_D = 80$		Overall	
	error	rank	error	rank	error	rank	error	rank
Atom-delay	12.2	2	7.9	1	17.1	2	12.4	1
Tyworth	14.0	3	20.1	3	23.2	3	19.1	4
Order statistics	15.0	4	17.3	2	23.4	4	18.6	3
Normal LTD	8.3	1	23.7	4	16.4	1	16.1	2
Gamma LTD	18.7	5	24.0	5	27.3	5	23.3	5
Weibull LTD	25.5	7	26.2	6	29.9	7	27.2	7
Exponential LTD	65.6	10	59.8	10	61.3	10	62.2	10
Beta LTD	20.9	6	27.2	7	29.1	6	25.7	6
Fixed LT	62.6	9	58.4	9	59.0	9	60.0	9
SD LTD	40.1	8	36.8	8	39.6	8	38.8	8

Table 6: Average absolute percentage errors and ranks of decision rules for $P_1 = 0.95$

From Tables 5 and 6 regarding the results of the P_1 measure, it can be concluded that the atom-delay, the normal, Tyworth and order statistics approaches are the four best approaches in terms of average absolute error for each of the three σ_D-values, with the atom-delay approach being superior in terms of the overall rank.

The fixed lead time approach and the exponential approach are both very poor in terms of estimating the reorder level. The gamma, beta and Weibull approaches have rank 5, 6 and 7 (interchanging depending on σ_D). However, there is a big difference between the results of $L_0 = 0$ and $L_0 = 4$. For $L_0 = 4$, the Weibull, gamma and beta distributions seem to be very good approximations of the lead time demand since the proposed reorder levels are close to the benchmark.

For $\sigma_D = 30$ (for C(v3) where $L_0 = 4$), the histogram of the lead time demand is depicted in Figure 3 along with the normal, Weibull and gamma density functions having the same mean and variance as the empirical data.

The density functions are not accepted by goodness of fit tests (Pearson's Chi-square and Kolmogorov-Smirnov), however as seen from the figure they all represent good approximations. Moreover, the parameters of the Weibull and gamma are so high that they are converging to the normal distribution.

There is not much difference between the relative rank over the three σ_D-values, so the variability of periodic demand does not appear to have any influence on which techniques are suitable.

Overall and average absolute errors and ranks of approaches for the P_2 measure can be seen in Table 8 below. The gamma distribution approach is omitted since it is only evaluated for $L_0 = 4$.

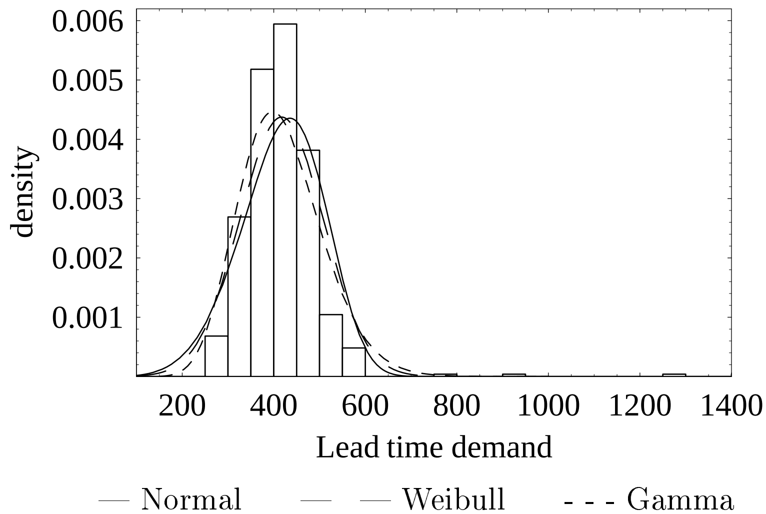

Figure 3: Empirical densities and Weibull, gamma and normal density functions

Sample	A		B		C(v1)		C(v2)		C(v3)	
L_0	0		0		0		0		4	
p_0	0.44	%	0.69	%	0.74	%	0.9	%	0.9	%
	s	error	s	error	s	error	s	error	s	error
$\sigma_D = 30$										
Benchmark	1053		386		660		67		497	
Atom-delay	1057	0.4	383	-0.8	696	5.4	-4	-106.0	441	-11.3
Tyworth	2447	132.4	374	-3.1	1102	67.0	-44	-165.7	439	-11.7
Order statistics	2369	125.0	363	-6.0	1066	61.5	-5	-107.5	437	-12.1
Normal LTD	1024	-2.8	233	-39.6	494	-25.2	39	-41.8	457	-8.0
Gamma LTD	NA		NA		NA		NA		433	-12.9
Weibull LTD	3562	238.3	1087	181.6	2759	318.0	0	-100.0	538	8.2
Exponential LTD	577	-45.2	95	-75.4	160	-75.8	-4	-106.0	1268	155.1
Beta LTD	244	-76.8	62	-83.9	88	-86.7	0	-100.0	416	-16.3
Fixed LT	204	-80.6	60	-84.5	82	-87.6	8	-88.1	384	-22.7
Moon&Choi	4125	291.7	482	24.9	1501	127.4	67	0.0	501	0.8
$\sigma_D = 50$										
Benchmark	1073		413		662		85		548	
Atom-delay	1105	3.0	468	13.3	691	4.4	122	43.5	575	4.9
Tyworth	2406	124.2	376	-9.0	1050	58.6	-49	-157.6	485	-11.5
Order statistics	2446	128.0	389	-5.8	1099	66.0	-4	-104.7	481	-12.2
Normal LTD	1045	-2.6	250	-39.5	513	-22.5	34	-60.0	490	-10.6
Gamma LTD	NA		NA		NA		NA		447	-18.4
Weibull LTD	3685	243.4	1147	177.7	2801	323.1	0	-100.0	585	6.8
Exponential LTD	580	-45.9	106	-74.3	171	-74.2	-4	-104.7	1284	134.3
Beta LTD	251	-76.6	67	-83.8	93	-86.0	0	-100.0	429	-21.7
Fixed LT	222	-79.3	70	-83.1	94	-85.8	13	-84.7	415	-24.3
Moon&Choi	4282	299.1	531	28.6	1576	138.1	64	-24.7	571	4.2
$\sigma_D = 80$										
Benchmark	1135		519		798		121		691	
Atom-delay	1420	25.1	535	3.1	936	17.3	0	-100.0	609	-11.9
Tyworth	3036	167.5	546	5.2	1344	68.4	-14	-111.6	602	-12.9
Order statistics	2839	150.1	508	-2.1	1279	60.3	0	-100.0	615	-11.0
Normal LTD	1264	11.4	302	-41.8	596	-25.3	74	-38.8	624	-9.7
Gamma LTD	NA		NA		NA		NA		533	-22.9
Weibull LTD	4321	280.7	1360	162.0	3173	297.6	0	-100.0	745	7.8
Exponential LTD	720	-36.6	130	-75.0	205	-74.3	0	-100.0	1544	123.4
Beta LTD	296	-73.9	78	-85.0	106	-86.7	0	-100.0	498	-27.9
Fixed LT	297	-73.8	97	-81.3	129	-83.8	23	-81.0	510	-26.2
Moon&Choi	5784	409.6	688	32.6	1965	146.2	1137	839.7	831	20.3

Table 7: Reorder levels and percentage errors of decision rules for $P_2 = 0.98$

	$\sigma_D = 30$		$\sigma_D = 50$		$\sigma_D = 80$		Overall	
	error	rank	error	rank	error	rank	error	rank
Atom-delay	24.8	2	13.8	1	31.5	2	23.4	1
Tyworth	76.0	6	72.2	5	73.1	5	73.8	6
Order statistics	62.4	3	63.3	3	64.7	3	62.5	3
Normal LTD	23.5	1	27.0	2	25.4	1	25.3	2
Weibull LTD	169.2	9	170.2	9	169.6	8	169.7	9
Exponential LTD	91.5	8	86.7	7	81.9	7	86.7	7
Beta LTD	72.7	4	73.6	6	74.7	6	73.7	5
Fixed LT	72.7	4	71.4	4	69.2	4	71.1	4
Moon&Choi	89.0	7	98.9	8	289.7	9	159.2	8

Table 8: Average absolute percentage errors and ranks of decision rules for $P_2 = 0.98$

From Tables 7 and 8 regarding the results of the P_2 measure, it can be concluded that the atom-delay approach and the normal distribution approach are by far the best approaches in terms of the average and overall absolute errors.

In general, the remaining approaches all perform very badly, and there are big differences between the approaches. The approach of Tyworth and order statistics often give similar results; however, they either overestimate or underestimate the true reorder level by a great amount. The Weibull approach gives high errors for $L_0 = 0$. However, for $L_0 = 4$ Weibull gives estimates very close to the benchmark and so does the Moon&Choi approach. The advantage of these two approaches (still for $L_0 = 4$) is that they overestimate the reorder level slightly ensuring that the target service level is indeed met, whereas the other approaches underestimate the reorder level leading to too low actual service levels.

For the P_2 measure the estimated reorder levels of C(v2) ($p_0 = 0.90$ and $L_0 = 0$) are quite special, since most approaches estimate the reorder level to be either zero or negative, whereas the benchmark is positive. In general there is a big difference in the accuracy of the reorder level estimation depending on whether L_0 is zero or 4. Most of the approaches do not perform well for L_0, but the approaches in general perform well for $L_0 = 4$. The question is, whether it is reasonable to assume $L_0 = 0$ at all, since the empirical data are based on the handling time of the order and is not including the actual transportation time to the customer.

The general conclusion from this comparison study is that the atom-delay approach is very suitable for both the P_1 service measure and the P_2 service measure and for both $L_0 = 0$ and $L_0 > 0$. Also, the normal distribution approach in general performs very well, both for the P_1 and the P_2 service measure. When $L_0 = 4$, the Weibull distribution approach produces very accurate results, especially for the P_2 service measure. The beta and the gamma distribution approaches are also quite accurate for $L_0 = 4$, especially for the P_1 service measure. The Tyworth approach and the order statistics approach perform reasonably well, especially for the P_1 service measure. These conclusions do not appear to depend on the demand variability, since the pattern is the same for different levels of σ_D.

There has been some discussion as to whether focus should be on reducing demand variability or lead time variability; see for instance [25, 6, 1, 9]. From this comparison study, however, by comparing the benchmark value of the reorder level (see for instance Table 9 in Appendix A), it can be seen that increasing σ_D, and thereby increasing the demand variability since the mean is fixed, the reorder level increases only by a small amount. However, decreasing p_0 (thus increasing the risk of a delay and thereby the mean and variance of lead time) increases the reorder level by a very big amount. Since the coefficient of variation of lead time actually decreases for decreasing p_0, the increase in mean lead time is actually the main cause of the increased reorder level. No firm conclusions can therefore be made as to where focus should be, but the issue of variability is important and needs to be further analysed in future research.

7 Summary and conclusions

In this paper a new atom-delay distribution approach has been proposed to modelling lead times. Hence, the lead time is assumed to take a fixed value, L_0, with a certain probability, p_0, and some delay with probability $1 - p_0$. Based on this atom-delay distribution for *lead time*, a new compound *lead time demand* distribution approach has been developed for determining reorder levels. For normally distributed demand and exponentially distributed delays this compound lead time demand distribution was developed. For $L_0 = 0$, the decision rules for the reorder level are relatively simple. For $L_0 > 0$ the computations are rather complex, but solvable with trial-and-error.

A number of alternative decision rules were presented that were either based on assumptions of the lead time demand distribution or based on distribution free approaches. These approaches were evaluated through a comparison study, where lead time demands and exact reorder levels were generated through simulation. Lead time demands were generated based on normally distributed demand and empirical lead time data. An (s, Q)-policy was used during the simulation.

From the comparison study it was concluded that the fixed lead time approach was inadequate to estimate the reorder level even for high values of p_0. Also, modelling lead time demand as exponential was a poor approximation for both service measures.

The atom-delay proved very well for both the P_1 and the P_2 service measures. Determining the reorder level by assuming normally distributed lead time demand was also a reasonable approach for both service measures. The distribution free approaches of Tyworth's procedure and order statistics performed very well for the P_1 service measure; however, they did not perform well for the P_2 service measure by either underestimating or overestimating the reorder level by a great amount.

It was also concluded that the demand variability did not have much effect on the relative performance of the approaches; however, it was quite important whether L_0 was equal to or greater than zero. For $L_0 = 0$, the gamma, beta and Weibull distributions did not provide good estimations of the reorder level, but for $L_0 = 4$ they performed quite well. For $L_0 = 4$, especially the gamma distribution under the P_1 service measure and the Weibull distribution under the P_2 service measure produced accurate estimates of the reorder level. In general, none of the approaches performed well under P_2, when $L_0 = 0$.

Hence, the atom-delay distribution and the normal distribution can be used for both the P_1 and P_2 service measures and for both $L_0 = 0$ and $L_0 > 0$. However, when $L_0 > 0$ and the P_2 service measure is used, the Weibull or Moon&Choi approaches are more accurate. To some extent, these results suggest that the shape of the lead time demand distribution indeed plays an important role. However, the two-parameter distributions, such as the gamma and the Weibull, produce reorder levels that are more accurate than the beta distribution. Unfortunately, the Schmeiser-Deutch distribution was subject to big sensitivity, so it is not possible to conclude whether this approach could perform better than the gamma or Weibull. For most cases also the normal distribution approach performs well, questioning the advantage of using the more complex computations of many of the alternative proposed procedures.

Improvements to the comparison study comprise the use of maximum likelihood estimation to estimate the distribution parameters of the lead time demand distribution and the development of computational (implementable) procedures for the atom-delay approach when $L_0 > 0$. Also, the run length in the current simulation is only 5000 periods due to the use of large variable structures, which use the maximum storage capacity in the simulation program. Even though the benchmark value of the reorder level is chosen to be representative among 10 replications, a higher run length may give more accurate estimates of the long run exact reorder level. By some extra programming effort this storage capacity problem could be overcome. However, for the current study it is evaluated that the reorder level is representative among the replications, since the average coefficient of variation of the reorder level is below 0.1. Another extension would be to develop the atom-delay approach for other lead time distributions than the exponential.

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Appendix A Simulated data for the 15 scenarios

In this appendix data regarding mean and standard deviation of the simulated processes are given for each of the 15 scenarios. Also, the simulated reorder level for the P_1 service measure is presented, which serves as a benchmark for comparison. Additional notation comprises μ_t, which denotes the mean delay conditional on a positive delay.

Sample	A	B	C(v1)	C(v2)	C(v3)
L_0	0	0	0	0	4
p_0	0.44	0.69	0.74	0.9	0.9
σ_D=30					
Benchmark, $s(P_1)$	1125	449	698	160	593
μ_D	99.614	99.614	99.614	99.614	99.614
σ_D^2	883.3513	883.3513	883.3513	883.3513	883.3513
μ_L	2.2582	0.7234	0.9631	0.1352	4.1352
σ_L^2	29.2407	3.2206	10.515	0.3833	0.3833
$CV(L)$	2.3945917	2.4807894	3.3669236	4.5792327	0.1497176
μ_t	4.1779	2.6154	4.0645	1.5714	1.5714
μ_{LTD}	234.33772	73.359738	99.62933	15.485319	417.21708
σ_{LTD}^2	312853.39	34320.011	113754.17	5754.5176	8321.0808
σ_D=50					
Benchmark, $s(P_1)$	1138	575	859	211	548
μ_D	104.042	104.042	104.042	104.042	104.042
σ_D^2	2263.3114	2263.3114	2263.3114	2263.3114	2263.3114
μ_L	2.2209	0.8715	1.0804	0.3059	4.3059
σ_L^2	27.7384	4.4163	10.0426	3.13	3.13
$CV(L)$	2.3714376	2.4113591	2.9331786	5.783526	0.4108736
μ_t	4.1625	2.8544	3.894	2.9259	2.9259
μ_{LTD}	235.21197	88.775464	115.99281	30.55414	444.41455
σ_{LTD}^2	325377.82	48144.789	119267.5	36175.248	44734.88
σ_D=80					
Benchmark, $s(P_1)$	1185	611	974	222	801
μ_D	115.7541	115.7541	115.7541	115.7541	115.7541
σ_D^2	4406.4549	4406.4549	4406.4549	4406.4549	4406.4549
μ_L	2.3739	0.7407	0.9965	0.1534	4.1534
σ_L^2	31.6591	3.4689	10.6808	0.5179	0.5179
$CV(L)$	2.3702102	2.514511	3.279628	4.6913474	0.1732683
μ_t	4.4451	2.7391	4.2183	1.7193	1.7193
μ_{LTD}	274.66022	87.904419	116.45155	19.82051	484.55519
σ_{LTD}^2	442384.59	49584.634	149486.51	10938.103	29284.836

Table 9: Mean and standard deviations of simulated processes

Appendix B Examples of numerical computations

In this appendix the numerical computations for determining reorder points of the various decision rules in the comparison study are illustrated in detail for one combination of experimental factors.

The combination of experimental factors is:
$P_1 = 0.95$, $\sigma_D = 30$, $p_0 = 0.44$, i.e. $L_0 = 0$,
and the empirical lead time data are for sample A.

From Table 9 on this scenario we have:

$$\begin{array}{lcl} \mu_D & = & 99.614 \\ \sigma_D^2 & = & 883.3513 \\ \mu_L & = & 2.2582 \\ \sigma_L^2 & = & 29.2407 \\ CV(L) & = & 2.3945917 \\ \mu_t & = & 4.1779 \\ \mu_{LTD} & = & 234.33772 \\ \sigma_{LTD}^2 & = & 312853.39 \\ (\sigma_{LTD} & = & 559.33) \end{array}$$

The atom-delay approach (exponential delay)

Data:
$\mu_D = 99.614$
$\sigma_D^2 = 883.3513$
$\lambda = 4.1779$ (mean delay)
$p_0 = 0.4332$

Then,
$\theta = 101.714$
$\omega = 0.00237775$
$\alpha = 0.95$ (target cycle service level)

Then,

$$s = \frac{1}{0.00237775} \ln\left(\frac{1 - 0.4332}{(1 - 0.95) * 4.1779 * 101.714 * 0.00237775^2}\right) = 1016.76.$$

Tyworth's procedure

Data:
$\mu_D = 99.614$, $\sigma_D = 29.721226$.

Using, trial-and-error: $s = 1194.3$, leading to the following computations in a spreadsheet:

L_i	p_i	μ_i	σ_i	$k_i(s)$	$\phi(k)$	$\Phi(k)$	$\Phi(k)p_i$	G_u	$G_u\sigma_i p_i$
0	0.433	0	0	0	0.398942	1	0.4332	0.398942	0
1	0.365	100	30	36.82	1.70E-295	1	0.3648	1.70E-295	1.84E-294
2	0.028	199	42	23.67	7.91E-123	1	0.028	7.91E-123	9.31E-123
3	0.014	299	51	17.39	7.89E-67	1	0.014	7.89E-67	5.69E-67
4	0.02	398	59	13.39	4.75E-40	1	0.0196	4.75E-40	5.54E-40
5	0.013	498	66	10.48	5.88E-25	1	0.0126	5.88E-25	4.92E-25
6	0.016	598	73	8.195	1.04E-15	1	0.0162	1.31E-16	1.55E-16
7	0.018	697	79	6.32	8.44E-10	1	0.0182	1.61E-11	2.30E-11
8	0.022	797	84	4.727	5.60E-06	1	0.0218	2.16E-07	3.97E-07
9	0.01	897	89	3.34	0.00151	0.99958	0.0096	0.000109	9.35E-05
10	0.002	996	94	2.108	0.043215	0.9825	0.00197	0.00632	0.001188
11	0.008	1096	99	1	0.24204	0.84128	0.00656	0.083361	0.064094
12	0.004	1195	103	-0.01	0.398921	0.49586	0.00198	0.40415	0.166441
13	0.008	1295	107	-0.94	0.256583	0.17373	0.00146	1.032896	0.929767
14	0	1395	111	-1.8	0.078792	0.03584	0	1.815351	0
15	0.006	1494	115	-2.61	0.013393	0.00459	2.94E-05	2.606864	1.920484
16	0.004	1594	119	-3.36	0.001408	0.00039	1.63E-06	3.360696	1.678051
17	0	1693	123	-4.07	9.96E-05	2.32E-05	0	4.073146	0
18	0	1793	126	-4.75	5.07E-06	1.03E-06	0	4.748364	0
19	0.002	1893	130	-5.39	1.95E-07	3.52E-08	7.74E-11	5.39063	1.536405
20	0.002	1992	133	-6	5.95E-09	9.69E-10	1.94E-12	6.00358	1.59596
21	0.002	2092	136	-6.59	1.48E-10	2.21E-11	4.85E-14	6.590275	1.974707
22	0	2192	139	-7.15	3.09E-12	4.26E-13	0	7.153321	0
23	0	2291	143	-7.69	5.54E-14	7.11E-15	0	7.694945	0
24	0	2391	146	-8.22	8.69E-16	1.11E-16	0	8.217073	0
25	0.002	2490	149	-8.72	1.21E-17	0	0	8.721376	2.33289
26	0	2590	152	-9.21	1.53E-19	0	0	9.209318	0
27	0.002	2690	154	-9.68	1.76E-21	0	0	9.682184	2.6915
...	...	...	...	...	...	...	...	...	...
99	0	9862	296	-29.3	1.15E-187	0	0	29.30953	0
100	0	9961	297	-29.5	4.54E-190	0	0	29.49777	0
						$\alpha =$	0.95	$B(s) =$	58.34
								$\beta =$	0.94

Order Statistics

First, the observed data on lead time demand are sorted in increasing order:

i	$x_{(i)}$
1	0
2	0
3	0
4	0
5	0
6	0
...	...
215	0
216	0
217	0
218	19.1
219	35.7
220	41.6
221	45.3
222	48.7
223	49.5
224	51.2
225	52.3
...	...
473	1113.3
474	1183.9
475	1208.4
476	1224.6
...	...
496	3569.8
497	3632.1
498	4010.8

Hence, $n = 498$. With $\alpha = 0.95$, $(n+1)\alpha = 499 * 0.95 = 474.05$

Therefore, $r = 474$ and $w = 0.05$. Then, the reorder level is:

$s = 0.95 * x_{(474)} + 0.05 * x_{(475)} = 0.95 * 1183.9 + 0.05 * 1208.4 = 1185.2.$

Normal LTD

Data:
$\mu_{LTD} = 234.34$
$\sigma_{LTD} = 559.33$
$k = \Phi^{-1}(0.95) = 1.6449$

Then, $s = 234.34 + 1.6449 * 559.33 = 1154.4$.

Gamma LTD

Data:
$\mu_{LTD} = 234.34$
$\sigma_{LTD} = 559.33$

Then,
$\lambda_1 = 0.175527$ (shape)
$\lambda_2 = 1335.05$ (scale)

Determine s so that, $0.95 = I_s(\frac{s}{\lambda_2\sqrt{\lambda_1}}, \lambda_1 - 1)$. By computing the incomplete gamma function ratio in *Mathematica*®, s is found by trial-and-error. Hence, $s = 1247.04$.

Weibull LTD

Data:
$\mu_{LTD} = 234.34$
$\sigma_{LTD} = 559.33$
$\mu_{LTD}/\sigma_{LTD} = 0.419$

The parameter c is tabulated as follows. For various values of c, $f(c) = \frac{\Gamma(1+1/c)}{\sqrt{\Gamma(1+2/c)-\Gamma(1+1/c)^2}}$ is calculated. c is then determined as the value where $\mu_{LTD}/\sigma_{LTD} = f(c)$. I.e. find the value of c where $f(c) = 0.429$.

c	$f(c)$	c	$f(c)$	c	$f(c)$
0.1	0.0023	1.1	1.0986	2.1	1.9988
0.2	0.0631	1.2	1.1949	2.2	2.0841
0.3	0.1849	1.3	1.2891	2.3	2.1688
0.4	0.3184	1.4	1.3817	2.4	2.2531
0.5	0.4472	1.5	1.4728	2.5	2.337
0.6	0.5688	1.6	1.5627	2.6	2.4205
0.7	0.6838	1.7	1.6516	2.7	2.5036
0.8	0.7933	1.8	1.7395	2.8	2.5865
0.9	0.8984	1.9	1.8266	2.9	2.6691
1	1	2	1.9131	3	2.7514

With more detailed increments of c and interpolation, c is estimated to 0.4776. Then b is calculated from $b = \mu_{LTD}/\Gamma(1 + 1/c) = 107.26$. Finally, the reorder point is calculated:

$s = 107.26 * (\ln(1/0.05))^{1/0.4776} = 1067.$

Exponential LTD

Data:
$\lambda = \mu_{LTD} = 234.34$

Then, $s = 234.34 * \ln(1/0.05) = 702.1.$

Beta LTD

Data:
$a = 0$ (minimum observed lead time demand)
$b = 4011$ (maximum observed lead time demand)
$\mu_{LTD} = 234.34$
$\sigma^2_{LTD} = 312853$
$\sigma_{LTD} = 559.33$

Then,
$\mu_s = 0.0584$
$\sigma_s = 0.13945$
$p = 0.1068$
$q = 1.722$

Determine s so that, $0.95 = I_r(p, q)$, where $r = \frac{s-a}{b-a}$. By computing the incomplete beta function ratio in *Mathematica*®, s is found by trial-and-error. Hence, $s = 1496.65$.

The fixed lead time approach

Data:
$\mu_D = 99.614$
$\sigma^2_D = 883.351$
$\mu_L = 2.2582$

Then,
$\mu_{LTD} = \mu_D \mu_L = 224.948$
$\sigma_{LTD} = \sqrt{\mu_L}\sigma_D = 44.663$
$k = 1.64485$

Hence, $s = 224.948 + 1.64 * 44.663 = 298.4.$

Schmeiser-Deutch LTD

Data:
$\mu_{LTD} = 234.34$
$\sigma^2_{LTD} = 312853$
$\mu_3 = 728,133,103$

$\mu_4 = 186,301,203,496$
$\alpha_3 = 4.161$
$\alpha_4 = 22.337$
$\omega_1 = 0.9454$
$\omega_2 = 0.0448$

From a figure in Kottas and Lau [16], c and d are read off:
$c = 10$
$d = 0.3$

Then, $p = 11$, $q = 21$, $a = 26.359$ and $b = 115710$

and finally, $s = 26.359 + 115710 * (0.95 - 0.3)^{10} = 1584.1$.