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Empirical Rationality in the Stock Market

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Abstract

This paper approximation errors are introduced in a Lucas (1978)-type model to reflect model uncertainty. The purpose is twofold. First, the rational investor is allowed to take model uncertainty into account when asset prices are determined. Second, the statistical degeneracy, common to most structural models, is broken and maximum likelihood inference made possible. The model is estimated using U.S. stock data. The equilibrium price is seriously affected by the existence of approximation errors and the descriptive and normative properties are greatly improved. This suggest that investors do not and should not ignore approximation errors.

Keywords: Approximation errors, rationality, structural estimation, risk premium, asset pricing.

1. Introduction

Once the state variabels x_t of a structural economic model are known, the model usually provide optimal equilibrium prices and decisions $d(x_t)$ which are singlevalued functions of the state.[1] This property is unfortunate if the model is meant to have descriptive value. Since all structural models are simplifications, their singlevalued predictions will almost never match the empirical data, $d_t \neq d(x_t)$. Two related problems arise. First, there is a strong tradition of modeling rational agents as if they believe the theoretical model to be an exact description of the economic reality. Thus rational agents are implicitly assumed to ignore the difference between their empirical environment and the theory, although there seems to be little a priori justification for this assumption. Secondly, since the model assigns no likelihood to the empirical data, the researcher is left with a statistically degenerated model where standard maximum likelihood inference is not possible.

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[1] In this paper, a "structural economic model" is an explicit description of how rational utility maximizing agents determine expectations, decisions, and equilibrium prices from their information on the state of the economy.

The two problems are entwined. Due to the symmetry assumptions of the rational expectation tradition, the researcher and the rational agent ought to believe the same model. If the researcher breaks the statistical degeneracy by adding error terms to the model, the rational agent should do the same. And vice versa: If model uncertainty is taken into account by the agent, this fact must also be reflected by the researchers model. Nevertheless, the literature has addressed the two problems separately so far. The model uncertainty literature mainly focuses on how agents react, but the empirical validity of the implications of the theoretical model is not questioned. Likewise, the literature that confront structural model with empirical data breaks the statistical degeneracy while insisting that rational agents trust the model unconditionally.

This paper treats the two problems in a unified setting. The hope of finding the *true* structural model is abandoned. Instead a *useful* structural model is considered. Neither the agents nor the researcher believe the useful model to be true. They recognize that approximation errors on behalf of the model separates the singlevalued implications of the model from empirical data. The approximation errors break the statistical degeneracy and allow the researcher to apply maximum likelihood estimation whereas the agent is allowed to take model uncertainty optimal into account when decisions are made.

The effect of approximation errors on decisions and prices can be significant. This paper estimates a Lucas (1978)-type asset pricing model and show the descriptive and the normative importance of including approximation errors when pricing U.S. stocks. The effect on the equilibrium stock price is decomposed into a *predictability effect* and a *risk premium effect*. Each adress important asset pricing puzzles.

The predictability effect is due to the empirical persistence of the approximation errors in the model. Observed errors, reflecting mispricing today, predict future errors. When this fact is taken into account by investors, the equilibrium stock price gets very sensitive to observed approximation errors. As a result, the distance between the observed price and the equilibrium price is reduced with 90% compared to traditional models without approximation errors, see Shiller (1981) and Grossman and Shiller (1981). The *stock market volatility puzzle* is therefore significantly reduced.

The risk premium effect arises when agents realize that asset investments are more risky when approximation errors are present. Although *equity premium puzzle* of Mehra and Prescott (1985) is not resolved, the empirical analysis show that the risk premium of U.S. stocks is approximately doubled for the Lucas (1978)-type model considered.

The normative analysis show that if real U.S. investors believe a Lucas (1978)-type asset pricing model without approximation errors they should have followed a very aggressive investment policy during certain periods of the past where the empirical price differed significantly from the fundamentals of the model. However, significant and persistent price deviations suggest that such an investment policy has not been used. The predictability- and the risk-effect might explain why. The expected return of such endeavors is greatly reduced and the risk increased once approximation errors are recognized. As a result a much less aggressive investment strategy should be followed by the rational investor.

The rest of the paper is organized as follows. Section 2 review existing approaches to model uncertainty and methods of breaking the statistical degeneracy. Section 3 describes a simple version of the asset pricing model of Lucas (1978) and argue that the model is statistically degenerated. Section 4 adds approximation errors to the asset pricing model and re-solves the rational agents investment problem taking the approximation errors into account. Section 5 estimates the model using U.S. stock data. The descriptive and normative importance of including approximation errors is shown. Section 6 concludes.

2. Existing Approaches

2.1. Model Uncertainty

Knight (1921) is the first to address the question of how economic agents react to shortcomings of theoretical models. Variations in the empirical environment are separated into *risk* and *(Knightian) uncertainty.*[2] Risk refers to events to which a theoretical model assigns well-defined probabilities, whereas uncertainty refers to events to which no objective probabilities can be assigned. The question of whether or not this distinction should have methodological implications, divides the literature in two.

One branch of the literature builds on Ellsberg (1961) who finds that people strictly prefer situations with less uncertainty with respect to the appropriate theoretical model. Even when they have no reason to expect a different distribution of utility. Since these findings are inconsistent with Savage (1954), this *Ellsberg paradox* suggests a methodological treatment of model uncertainty outside the traditional paradigm. Along these lines, Gilboa and Schmeidler (1989) suggest that when model uncertainty is present, agents consider a set of possible models and expect the worst model to apply. This *least favorable prior* approach to model uncertainty have been applied by, for instance, Epstein and Wang (1994), Hansen, Sargent, and Tallarini (1999), and Hansen and Sargent (2000). Although this research outside the traditional paradigm of Savage (1954) might have significant descriptive value, the normative value of the approach can be questioned, see Sims (2001).

Another interpretation of Knight (1921) is found in e.g. LeRoy and Singell (1987) and Hirshleifer and Riley (1992, p. 9), see also Arrow (1951). This interpretation follows Savage (1954) and claims that when no objective probabilities can be assigned to model uncertainty, subjective probabilities are formed and treated as objective probabilities. Confidence in the subjective probabilities is recognized to play a role but usually dealt with in the traditional framework of Savage using a Bayesian approach. The true model is assumed to belong to a specified set of possible models and each model is assigned a subjective prior which is updated when empirical observations are made, see e.g. Draper (1995) and Hansen and

[2] The term *model uncertainty* will often be used below instead of the term (Knightian) uncertainty

Sargent (2000).

Both branches investigates how model uncertainty affects the optimal decisions of the agent $\tilde{d}(x_t)$ but they do not adress the researchers problem of statistical degeneracy. Regardless whether the decisions are based on the least favorable model or Bayesian priors over a set of models, the optimal decisions and the equilibrium prices will be singlevalued. Therefore, statistical degeneracy,

$$d_t \neq \tilde{d}(x_t), \tag{1}$$

will prevail for both approaches.

2.2. Statistical Degeneracy

At first, the statistical degeneracy of $d(x_t)$ might seem as a minor problem. Following a pragmatic statistical approach an approximation error term z_t could be added to reflect the shortcomings of the theory

$$d_t = d(x_t) + z_t \tag{2}$$

If appropriate distributional assumptions are made with respect to z_t, this will break the statistical degeneracy and make maximum likelihood inference possible. However, the approach in (2) is inconsistent from a symmetry point of view since the optimal decisions are based on the $d_t = d(x_t)$- assumption. The literature has been reluctant to abandon this assumption in order to reestablish symmetry . Instead, other approaches have been used to break the statistical degeneracy.

The *measurement error* approach assumes that the observed data d_t and the true empirical values d_t^0 differ by a measurement error term, $d_t^0 = d_t + e_t$. At the same time, e_t is assumed to equal the difference between theory and observed data,

$$d_t + e_t = d(x_t)$$

Therefore, the theory match the true data, $d_t^0 = d(x_t)$, and the theoretical model can still be considered true by both researcher and agents. If distributional assumptions are made with respect to e_t, maximum likelihood inference can take place.[3] The literature on maximum likelihood estimation of structural macromodels has mainly used this approach, see Altuğ (1989), Watson (1993), McGrattan (1994), and others.

The *unobserved state variable* approach recognize that the singlevalued relationship from state variables to decision and price variables will be violated by data. To avoid this direct confrontation, models are only taken to data if the state variables, or a subset of these, are unobserved by the researcher,

$$d_t = d(?).$$

Contrary to the researcher, the agents is assumed to observed the state variables and their decisions will reflect this knowledge. Using the observed behavior, the

[3] A full measurement error analysis includes state variables measurement errors also.

state variables might then be identified by the researcher, if the decision function is invertible, $\hat{x}_t = d^{-1}(d_t)$. If distributional assumptions are made with respect to x_t, maximum likelihood inference can take place. The approach has been wildly used in estimations of structural micromodels, see Wolpin (1984), Miller (1984), Pakes (1986), Rust (1987), and others.

The *GMM-approach*, see Hansen (1982), is wildly used for estimation of models involving intertemporal optimization. However, the approach is applied to models where expectations with respect to the evolution of state variables x_t are not formed explicitly. As a consequence, an explicit structural relationship between x_t and d_t cannot be established. Thus, the GMM-approach is not applicable to the structural models considered here unless parts of the structure are ignored.

Both the measurement error approach and the unobserved state variable approach succeed in breaking the statistical degeneracy while maintaining the theoretical model as true from both the researchers and the agents' point of view. As a consequence, the approaches do not adress how model uncertainty affect the behaviour of the agents.

2.3. The Approximation Error Approach

This paper unifies the two approaches described by (1) and (2) in order to reestablish the symmetry property. Accepting model uncertainty and breaking the statistical degeneracy are treated as two sides of the same coin.

Both the researcher and the agent recognize approximation errors on behalf of the theoretical model. These are the reason for the deviation between theory and data. The approach differs from the one of (2) since the decisions of the agents are affected by the existence of approximation errors, $d(x_t) \neq \tilde{d}(x_t)$. However, the approach does not share the view of (1). Even if approximation errors are taken into account by the agents, the implications of the theory will still differ from empirical data with an approximation error:

$$d_t = \tilde{d}(x_t) + z_t. \tag{3}$$

Following Knight (1921), the $\tilde{d}(x_t)$-term should be interpreted as the risk-part of the empirical variation in d_t. That is, the part to which well-defined objective probabilities are assigned. The z_t-term, however, represents Knightian uncertainty since this part of the variation is not accounted for by a structural theoretical model.

With respect to the methodological consequences of this distinction, this paper follows the interpretations of LeRoy and Singell (1987) and stay in the traditional normative framework of Savage (1954). Therefore, subjective believes with respect to the distribution of z_t are formed by the researcher as well as the agents. Since these believes are treated as objective probabilities, the statistical degeneracy is broken and model uncertainty can be taken into account by the agents.

The symmetry assumption is central for closing the model with respect to expectations. If the researcher believe (3) subject to a distributional assumption with respect to z_t, the agents should derive $\tilde{d}(x_t)$ believing (3) as well as the same distributional assumptions as the researcher.

It might not be obvious why d_t is chosen by the agents when $\tilde{d}(x_t)$ is recommented by the structural model. The problem is especially troubling at the microlevel. However, a fundamental assumption of the approximation error approach is that agents consider the structural model as a useful tool, not the true description of the economic environment. The individual agent recognize that events, not described by the structural model, will take place and make the agent deliberately decide different from what is implied by the model.[4]

The approximation error approach has some advantages and disadvantages compared with existing approaches. Compared to the least favorable prior literature, the approximation error approach is kept in the traditional normative framework of Savage (1954) and avoid the criticism of Sims (2001). However, the normative property might also be a drawback from a descriptive point of view. Especially, the approach is not capable of addressing observations like the Ellsberg paradox.

Compared to the Bayesian approach four points can be made in favor of the approximation error approach. First of all, the approximation error approach is not statistically degenerated like the Bayesian approach. Secondly, priors are modeled by reduced form distributions and since the approximation errors are well-defined and observable, the distributional assumptions can be tested. This is not the case for priors of the Bayesian approach, which has been the traditional criticism of that approach. Third, the approximation error approach introduces misspecification while maintaining rational expectations. Especially, rational expectations are formed with respect to the impact of misspecification.[5] Finally, as shown below, the approach introduces model uncertainty in a very tractable manner from a technical point of view.

The main drawback compared to the Bayesian approach is that no learning takes place with respect to the appropriate model. However, learning with respect to the parameters might still take place.

Four points can also be raised in favor of the approximation error approach compared to the traditional methods of breaking the statistical degeneracy. First of all, the approximation error approach allow the rational agent to recognize the model uncertainty of structural models.

Secondly, the approach offers a consistent criteria for choosing between competing models. Both the measurement error and the unobserved state variable approach base the likelihood functions on unobserved variables which can only be identified by the true structural model. If two competing models are considered, two series of unobserved variables, two distributions, and two likelihood functions are available. However, no guidance for choosing the appropriate likelihood function is provided. Non-likelihood criteria such as minimizing the size of measurement errors or unobserved state variables do not solve the problem. If these variables are assumed to exist, it is not obvious why the model with the smallest measurement errors is more true than other models. Ultimately, why should a model implying

[4]The optimization error approach to statistical degeneracy assumes that the difference between the optimal and the actual decisions are unintended by the agents, see Rust (1994). The approach has received little attention in the literature. Beside the different interpretation of z_t, the approach should be implemented quite similar to the approximation error approach.

[5]Hansen and Sargent (2000) argue that model misspecification is absent in the Bayesian approach and that rational expectations excludes model misspecification.

measurement errors of size zero be preferred to other models when measurement errors are assumed to exist? Such problems do not arise when the approximation error approach is used. The series of approximation errors implied by different models are true conditional on the model they were determined with. Therefore, it is perfectly legitimate to choose the model with the smallest observed approximation errors.

Third, the basic assumptions underlying the traditional approaches do not seem valid in general. Although the unobserved state variable approach maintains the symmetry assumption with respect to the model, the approach breaks the symmetry with respect to information: The agents observe more than the researcher. While this asymmetry might be reasonable in a microeconomic setting, it is not generally acceptable. In a macroecomomic setting it seems implausible to assume that some macroeconomic state variables is observed by all others than the researcher. Moreover, the asymmetry is inconsistent if the researcher and the rational agent happens to be the same person. Furthermore, measurement errors are not a reasonable explanation when the data quality is high and the empirical fit is low which is the case for many financial models.

Finally, the approximation error approach also offers technical advantages. Normally, the measurement error approach cannot be applied to non-linear specifications, see Rust (1994). Moreover, to apply the unobserved state variable approach an invertability condition must be satisfied, see Pakes (1994). As shown below, such restrictions are not imposed on the approximation error approach.

3. A Theoretical Asset Pricing Model

Assume that a Lucas (1978)-type asset pricing model is the relevant theoretical model for the situation at hand. At time t a representative investor, or equivalently a number of identical investors, receives an endowment e which can be used for consumption c to gain utility u. However, the investor faces a tradeoff between consuming today and investments in a financial asset which pays stochastic dividends d in the future. To maximize the expected infinite horizon utility at time 0, the investor solves the problem

$$\begin{aligned} &\max_{w} E_0 \sum_{t=0}^{\infty} \beta^t u(c_t), \\ &s.t. \\ &c_t = e_t + d_t w_t + p_t(w_t - w_{t+1}), \end{aligned} \tag{4}$$

where d_0 and c_0 are given, w is the investors holding of the financial asset, p is the price of the asset and the time additive utility is discounted with a constant factor $0 < \beta < 1$. E_0 denotes expectations conditioned at information at time 0. The utility function is assumed to be of a standard *constant relative risk aversion*-type:

$$u(c_t) = \frac{c_t^{(1-\gamma)} - 1}{1-\gamma}$$

where $\gamma > 0$ is known as the risk aversion parameter. To simplify, the holdings w are assumed to be excess demand. Therefore, $w = 0$ and $c = e$ in equilibrium. Consumption and dividends are assumed to follow stationary exogenous Markov processes. Two alternative autoregressive specifications are considered below: One specification with additive shocks,

$$(5.A) \qquad \begin{aligned} d_t &= (1-\delta) + \delta d_{t-1} + \epsilon_t^d, \\ c_t &= (1-\theta) + \theta c_{t-1} + \epsilon_t^c, \end{aligned}$$

and one with multiplicative shocks,

$$(5.M) \qquad \begin{aligned} d_t &= d_{t-1}^{\delta} \exp(\epsilon_t^d), \\ c_t &= c_{t-1}^{\theta} \exp(\epsilon_t^c), \end{aligned}$$

where $|\delta| < 1$, $|\theta| < 1$, and $\epsilon_t = [\epsilon_t^d, \epsilon_t^c]$ is independent normal distributed with mean vector μ and covariance matrix Ω. Although not indicated by the notation, μ and Ω will differ between (5.A) and (5.M).

Since the additive specification allow negative values, problems arise since negative consumption is not allowed when $\gamma \geq 1$. However, given the parameter values estimated below, the probability of negative consumption is very small. In order to avoid further restrictions on γ, the consumption distribution will be truncated at zero. The two specifications each have advantages. The multiplicative specification in (5.M) offers closed form solutions for the equilibrium price whereas the additive specification in (5.A) offers an intuitive interpretation of the effects of model uncertainty in Section 4.

To solve for the equilibrium price at time 0, consider the Euler equation obtained from using first order conditions with respect to w_1 and the equilibrium condition $w = 0$,

$$(6) \qquad p(d_0, c_0) = E_0 \widetilde{\beta}\,(d_1 + p_1)\,, \quad \widetilde{\beta} \equiv \beta \frac{u'(c_1)}{u'(c_0)},$$

where d_0 and c_0 summarize all available information about the state of the economy at $t = 0$ due to the Markov property. To determine $p(d_0, c_0)$, only the investors expectations with respect to p_1 are left to be specified. Following the rational expectation tradition initiated by Muth (1961), p_1 is substituted with $p(d_1, c_1)$ to get a functional equation for the price function:

$$(7) \qquad \overset{\mathrm{T}}{p}(d_0, c_0) = E_0 \widetilde{\beta}\big(d_1 + \overset{\mathrm{T}}{p}(d_1, c_1)\big),$$

where $\overset{\mathrm{T}}{p}$ denote the price function derived using the theoretical model to form expectations.

Alternatively, $\overset{\mathrm{T}}{p}(d_1, c_1)$ might be replaced by the right hand side of the Euler equation derived from the first order condition with respect to w_2. This leaves $\overset{\mathrm{T}}{p}(d_0, c_0)$ as a function of p_2 or, when expectations are formed using the theoretical model, as a function of $\overset{\mathrm{T}}{p}(d_2, c_2)$. Applying this substitution repeatedly, and ruling out bobbles etc. gives

$$(8) \qquad \overset{\mathrm{T}}{p}(d_0, c_0) = E_0 \sum_{t=1}^{\infty} \widetilde{\beta}^t d_t, \quad \widetilde{\beta}^t \equiv \beta^t \frac{u'(c_t)}{u'(c_0)}.$$

The price equals the expected sum of dividends discounted with a stochastic discount factor $\widetilde{\beta}^t$ which depends on the marginal utility of consumption. If the multiplicative specification in (5.M) is assumed, (8) and the fact that $u'(c_t)d_t$ is then lognormal distributed offers a closed form solution:

$$(9) \qquad \overset{\mathrm{T}}{p}(d_0, c_0) = \sum_{t=1}^{\infty} \beta^t c_0^{-\gamma(\alpha^t - 1)} d_0^{\delta^t} \Lambda_t,$$

where Λ_t is independent of d_0 and c_0, see Appendix A. Due to the truncation of the consumption distribution, closed form solutions are ruled out when the additive specification in (5.A) is assumed. In that case, numerical procedures are used to determine the equilibrium price, see Appendix B.

The expression for the equilibrium price in (9) clearly illustrates the statistical degeneracy common to most structural models. If empirical values for p_0, d_0, and c_0 are observed, $\overset{\mathrm{T}}{p}(d_0, c_0)$ can easily be calculated for fixed parameters. But since $\overset{\mathrm{T}}{p}$ is a deterministic singlevalued function, the equilibrium price will never match the empirical counterpart p_0. Varying the parameters or choosing other functional forms might bring the model to match data for some observations but in general

$$(10) \qquad p_t \neq \overset{\mathrm{T}}{p}(d_t, c_t)$$

will prevail for most observations.[6] Since the model do not assign likelihood to situations like (10), the model is statistically degenerated and maximum likelihood inference not possible.

4. An Empirical Asset Pricing Model

Before proceeding, some practical terminology is introduced in order to distinguish between two groups of structural models.

Definition 1: A model that do not assigns likelihood to the empirical data is a *theoretical model.* Expectations formed using a theoretical model is called *theoretical expectations.* Behavior, which is optimal according to a theoretical model and the matching theoretical expectations, is said to reflect *theoretical rationality.*

The asset pricing model described in Section 3 is clearly theoretical and since $p_t = \overset{\mathrm{T}}{p}(d_t, c_t)$ is assumed, expectations are theoretical too. The equilibrium price in (8) therefore reflect theoretical rationality. The counterparts to theoretical models are empirical models:

[6] A trivial exception is the case where the number of free parameters equal or exceed the number of observations multiplied with the number of implications of the model.

DEFINITION 2: A model that assigns likelihood to the empirical data is an *empirical model.* Expectations formed using an empirical model is called *empirical expectations.* Behavior, which is optimal according to an empirical model and the matching empirical expectations, is said to reflect *empirical rationality.*

The assignment of likelihood to data is exactly what breaks the statistical degeneracy. Moreover, it seems like the weakest possible consistency requirement if a model must reflect believes with respect to the empirical environment. The next sections describe how the theoretical model of Section 3 might be used to set up a useful empirical model.

4.1. EMPIRICAL RATIONALITY, AN ADDITIVE SPECIFICATION

Assume that the model in (4) with the additive specifications of the state processes in (5.A) is a useful theoretical model. Yet, the implications of the model, here the equilibrium price $\overset{\mathrm{E}}{p}$, differ from the empirical asset price p_t with an approximation error term z,

$$p_t = \overset{\mathrm{E}}{p} + z_t, \tag{11}$$

where $\overset{\mathrm{E}}{p}$ is to be determined below. Parallel to the interpretation of (3), $\overset{\mathrm{E}}{p}$ is the implication of the objective theory whereas z_t reflects (Knightian) uncertainty. Let the subjective believes of both the researcher and the investors be described by

$$z_t = \zeta z_{t-1} + \epsilon_t^z, \tag{12}$$

where $|\zeta| < 1$ and $\epsilon_t = [\epsilon_t^d \ \epsilon_t^c \ \epsilon_t^z]^\top$ is independent joint normal distributed with a mean vector μ and covariance matrix Ω. Since likelihood is assigned to all rational values of z_t, this model is an empirical model for all finite values of p_t and $\overset{\mathrm{E}}{p}$. Finally, assume that the investor is aware of the approximation errors and use (11) and (12) to form *model uncertainty augmented* expectations about future prices. Thus empirical expectations are formed.

In order to determine the equilibrium price based on empirical rationality, notice that the Euler equation in (6) is still valid. However, when expectations with respect to next periods price is formed, p_1 should be substituted with $\overset{\mathrm{E}}{p} + z_t$ instead of $\overset{\mathrm{T}}{p}$. Therefore, (7) is replaced by

$$\overset{\mathrm{E}}{p}(d_0, c_0, z_0) = E_0 \widetilde{\beta}\big(d_1 + \overset{\mathrm{E}}{p}(d_1, c_1, z_1) + z_1\big). \tag{13}$$

Applying the usual repeated substitution procedure, but now using (11), gives

$$\begin{aligned} \overset{\mathrm{E}}{p}(d_0, c_0, z_0) &= E_0 \sum_{t=1}^{\infty} \widetilde{\beta}^t (d_t + z_t) \\ &= \overset{\mathrm{T}}{p}(d_0, c_0) + E_0 \sum_{t=1}^{\infty} \widetilde{\beta}^t z_t. \end{aligned} \tag{14}$$

Notice that $\overset{\mathrm{E}}{p}$ coinside with $\overset{\mathrm{T}}{p}$ if approximation errors are absent. Naturally, $\overset{\mathrm{E}}{p}$ will have to be solved numerically like $\overset{\mathrm{T}}{p}$ when the additive specification is assumed, see Appendix B.

The effect of the approximation errors is naturally decomposed into a *predictability effect* Δ_t and a *risk premium effect* π_t^z:

$$\overset{\mathrm{E}}{p}(d_0, c_0, z_0) - \overset{\mathrm{T}}{p}(d_0, c_0) = \underbrace{E_0 \sum_{t=1}^{\infty} \widetilde{\beta}^t E_0(z_t)}_{\Delta_0} + \underbrace{E_0 \sum_{t=1}^{\infty} \widetilde{\beta}^t \left(z_t - E_0(z_t)\right)}_{\pi_0^z}. \tag{15}$$

The predictability effect arise when $\zeta \neq 0$. Assume, for instance, $\zeta > 0$ and that the investor observes a higher price than predicted by the equilibrium price of the model, $z_0 > 0$. Then, positive approximation errors should also be expected in the next period, $E_0 z_1 > 0$. The predictability effect is illustrated by the Euler equation in (13) with z_1 replaced by $E_0 z_1$. Since the investor predicts empirical prices above next periods equilibrium price, he raises the current equilibrium price with $E_0 \widetilde{\beta} \zeta z_0$. However, next periods equilibrium price, and in fact all future prices, is also affected by higher approximation errors. Foreseeing the increase in future equilibrium prices, the investor raises the current equilibrium price even further. The effect is $E_0 \sum_{t=2}^{\infty} \widetilde{\beta}^t \zeta^t z_0$. The sum of this derived effect and the direct effect equals Δ_0 in (15). Notice, that in the case of constant consumption, the predictability simplifies to

$$\Delta_0 = \frac{\beta \zeta}{1 - \beta \zeta} z_0. \tag{16}$$

Thus for ζ- and β-values close to one, the predictability effect is significantly larger than the approximation error itself.

The definition of the approximation error risk premium in (15) parallel the definition of the *traditional risk premium* of the theoretical model:

$$\pi_0^d = E_0 \sum_{t=1}^{\infty} \widetilde{\beta}^t \left(d_t - E_0(d_t)\right).$$

Moreover, the wish to smooth consumption generates a *consumption risk premium*:

$$\pi_0^c = E_0 \sum_{t=1}^{\infty} \beta^t \frac{u'(c_t) - u'\left(E_0(c_t)\right)}{u'(c_0)} \left(E_0(d_t) + E_0(z_t)\right).$$

Notice that with these definitions, $\pi^d + \pi^c + \pi^z$ equals the difference between the equilibrium price and the *uncertainty equivalent* counterpart. Section 5 estimate and compare π^z, π^c, and π^d.

4.2. Empirical Rationality, a Multiplicative Specification

Again, let (4) be a relevant theoretical asset pricing model, but now d and c are assumed to follow the multiplicative processes defined in (5.M). Moreover, assume that the believes with respect to approximation errors are described by

$$p_t = \overset{\mathrm{E}}{p}\, z_t, \tag{17}$$

where the specification of the z-process corresponds to the d and c processes,

$$z_t = z_{t-1}^{\zeta} \exp \epsilon_t^z. \tag{18}$$

As before, $|\zeta| < 1$ and $\epsilon_t = [\epsilon_t^d \ \epsilon_t^c \ \epsilon_t^z]^\top$ is independent joint normal distributed. Finally, assume that empirical expectations are formed. Contrary to the model with additive specification, the model described by (4), (5.M), (17), and (18) do not assign likelihood to negative values of d, c, and z. However, it is still an empirical model when p_t and $\overset{\mathrm{E}}{p}$ takes positive and finite values.

As before, the Euler equation in (6) is still valid. However, once the uncertainty augmented expectation equation in (17) is used for substitution of p_1, the relevant Euler equation is changed to

$$\overset{\mathrm{E}}{p}(d_0, c_0, z_0) = E_0 \left[\widetilde{\beta}\big(d_1 + \overset{\mathrm{E}}{p}(d_1, c_1, z_1) z_1\big) \right].$$

Repeated substitution using (17) gives

$$\overset{\mathrm{E}}{p}(d_0, c_0, z_0) = E_0 \sum_{t=1}^{\infty} \widetilde{\beta}^t d_t z_{t-1} z_{t-2} \cdots z_1.$$

Notice, that $\overset{\mathrm{E}}{p}$ coinside with $\overset{\mathrm{T}}{p}$ if the approximation errors vanish. Due to the multiplicative specification and the fact that the asset price depends on all future dividends, a single approximation error will affect the importance of all succeeding dividends.

Even with approximation errors, the multiplicative specification offers a closed form solutions to the equilibrium price,

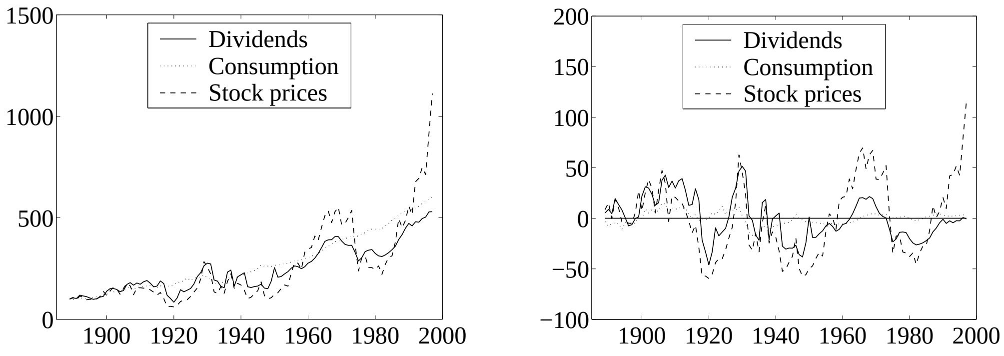
$$\overset{\mathrm{E}}{p}(d_0, c_0, z_0) = \sum_{t=1}^{\infty} \beta^t c_0^{-\gamma(\theta^t - 1)} d_0^{\delta^t} z_0^{\frac{\zeta - \zeta^t}{1-\zeta}} \Psi_t, \tag{19}$$

where Ψ_t is independent of d_0, c_0, and z_0, see Appendix A. The z_0-term in (19) clearly shows a predictability effect when $\zeta \neq 0$. Nevertheless, the effect of the approximation errors is not decomposed as easily as for the model with additive specification. The definitions of the different risk premiums can be found in Appendix A together with their closed from expressions.

Although the approximation error specification in (17) and (18) is stationary, the effect on the equilibrium price can be dramatic. If μ_z and σ_z^2 denote the mean and variance of ϵ^z, Appendix A shows that $\overset{\mathrm{E}}{p}$ is infinite when

$$\mu_z > \bar{\mu}_z \equiv -\tfrac{\sigma_z^2}{2(1-\zeta)}.$$

For $\mu_z \leq \bar{\mu}_z$, however, $\overset{\mathrm{E}}{p}$ is finite and well-behaved. Thus, the equilibrium price displays a serious discontinuity even for reasonable parameter values.



(a) Real series (1889 = 100).

(b) De-trended series (% from mean)

FIGURE 1: The empirical data
The dividends and the stock prices refer to the S&P Composite Stock Price Index provided by Robert J. Shiller from his Yale-homepage. Consumption is real per capita consumption of non-durables and services. Annual data, 1889–1997.

5. ESTIMATION

The introduction of approximation errors in Section 4 has broken the statistical degeneracy without violating the symmetry assumption. This section estimates the model both in a theoretical rationality version and an empirical rationality version. The results are compared in order to access the empirical importance of empirical rationality.

5.1. DATA

The data used for the empirical analysis are updated series of the yearly data for dividends, stockprices, and per capita consumption used in Shiller (1989). The consumption series have been updated using data from the Bureau of Economic Analysis. Figure 1(a) shows real value indexes for all three variables of the model. In order to work with finite state space, the series are de-trended with a constant growth rate of 1.60% per year. This is the growth rate of the per capita consumption during the estimation period. The result is seen in Figure 1(b). The mean of the de-trended d and c series are normalized to 1 and the price series are transformed accordingly in order to keep the average empirical price-dividend ratio at 22.8 fixed.

5.2. ESTIMATION OF THE EXOGENOUS PROCESSES d AND c

Since the d and c processes are exogenous to the model, it is possible to estimate their parameters δ, γ, and Ω independent of the asset pricing model in (4).[7] Table 1

[7]The values of μ are determined by restricting the unconditional expected values of the series to $E[d_t] = 1$ and $E[c_t] = 1$. For the additive specification, μ-values equal zero. For the multiplicative case, see Appendix A.

Table 1

Estimates of d and c process parameters

Specification		δ/θ	σ_d/σ_c	$\rho_{d,c}$	R^2	*Normality* *P−value*	DW
Additive	d	0.8740 (0.0485)	0.1145 (0.0078)	0.4063 (0.0816)	0.7087	38.5176 0.0000	1.7007
	c	0.8341 (0.0455)	0.0317 (0.0022)		0.7444	8.7012 0.0129	2.0768
Multiplic.	d	0.8652 (0.0499)	0.1179 (0.0080)	0.3611 (0.0850)	0.7090	33.4978 0.0000	1.6897
	c	0.8437 (0.0453)	0.0314 (0.0021)		0.7456	13.3207 0.0013	2.1021

Notes: σ_d and σ_c are standard deviations of ϵ^d and ϵ^c, $\rho_{d,c}$ is the correlation. Figures in brackets are standard deviations based on the hessian of the loglikelihood function. The normality test is a $\chi(2)$-distributed Jarque–Bera joint skewness and kurtosis test. DW is the Durbin–Watson test statistic for autocorrelation.

shows maximum likelihood estimates of the parameters in (5.A) and (5.M). The choice of specification seems to play a minor role for the estimates. Both dividends and consumption are significantly serial correlated and nearly 75% of total variation is explained by these simple specifications. For both specifications, the residuals display significantly more kurtosis than the normal distribution. The estimates in Table 1 will be taken as given when solving the price function and when estimating the other parameters of the model.

5.3. The Likelihood Function

In order to assess the effect of empirical rationality, the asset pricing model will be estimated in a theoretical rationality version as well as an empirical rationality version. Both versions are estimated for additive and multiplicative specifications. Consider first the two models based on empirical rationality analyzed in details in Section 4:

$$M_A^E \qquad p_t = \overset{\mathrm{E}}{p}(d_t, c_t, z_t) + z_t,$$

$$M_M^E \qquad p_t = \overset{\mathrm{E}}{p}(d_t, c_t, z_t) z_t,$$

where the subscript of M_A^E and M_M^E refer to additive/multiplicative specification and the superscript refer to empirical rationality. Their counterparts are

$$M_A^T \qquad p_t = \overset{\mathrm{T}}{p}(d_t, c_t) + z_t,$$

$$M_M^T \qquad p_t = \overset{\mathrm{T}}{p}(d_t, c_t) z_t,,$$

where the superscript indicate that the equilibrium price is based on a theoretical rationality. Thus even though the researcher believes M_A^T and M_M^T, which by the way are empirical models, the investor ignores z when the equilibrium price is determined.[8]

The likelihood of the price series p_t, $t = \{2, T\}$, is conditioned on the first observation p_1 and the free parameters $\omega = (\gamma, \beta, \zeta, \sigma_z, \rho_{d,z}, \rho_{c,z})$.[9] As mentioned, the observations of d_t and c_t, the parameters of Table 1 and the implied values of ϵ_d and ϵ_c are exogenously by assumption. The likelihood of the price series is given by

$$\begin{aligned}\ell(p_T, p_{T-1}, \cdots, p_2 | p_1, \omega) &= \sum_{t=2}^{T} \ell(p_t | p_{t-1}, \omega) \\ &= \sum_{t=2}^{T} \left(\frac{\partial M}{\partial z_t} \frac{\partial z_t}{\partial \epsilon_t^z} \right)^{-1} \ell(\epsilon_t^z | \omega),\end{aligned}$$

where $M \in \{M_A^T, M_M^T, M_A^E, M_M^E\}$ and the jacobian $\frac{\partial M}{\partial z} \frac{\partial z}{\partial \epsilon}$ is used for change of measure from ϵ to p.

5.4. Parameter Estimates

Panel A of Table 2 shows the estimation results. Notice that γ is imprecisely estimated and, based on a 95% likelihood ratio test, log-utility ($\gamma = 1$) cannot be rejected for any of the four models.

Panel B of Tabel 2 shows the estimation results with the log-utility condition imposed. The normality of z_t for the multiplicative specification is only accepted at a 1% significance level. In return, the precision of all estimates as well as the R^2-values are improved compared to Panel A. Since the restriction also improves the comparability of risk premiums across models, log-utility is imposed in the analysis below.

Except for σ_z, the parameter estimates in Panel B are constant across models. The approximation errors show high autocorrelation with estimates of ζ around 0.9. The discount factor estimates are between 0.96 and 0.965,[10] and the correlations between the shocks, $\rho_{d,z}$ and $\rho_{c,z}$, are significantly positive and approximately 0.5 and 0.35 respectively. Keeping the rationality type fixed, the difference between the σ_z-estimates for the additive and the multiplicative specification is approximately equal to the empirical price/dividend ratio of 22.8, which is what to expect.

The key result of Table 2 is the dramatic decrease in the size of approximation errors when empirical rationality is introduced. As a result, the estimates of σ_z are reduced with 85-90%. The reduction is accompanied by an equally dramatic increase in the R^2-values. Both changes are due to a major predictability effect.

[8] To get consistency from a symmetry point of view, z_t in M_A^T and M_M^T must be interpreted as a measurement error. Notice, however, that a full measurement error analysis is not possible, due to the non-linearity of the problem.

[9] For the additive specifications, μ_z is restricted to 0. For the multiplicative specifications, μ_z is restricted to $\bar{\mu}_z$, see Section 4

[10] Notice, that in order to relate to yearly returns the β-estimates should be corrected for the missing growth rate of 1.6%.

Table 2

Estimation results

	β	γ	ζ	σ_z	$\rho_{d,z}$	$\rho_{c,z}$	*Normality* *P–value*	DW	R^2	ℓ
				Panel A: All parameters free						
M_A^T	0.9637 (0.0077)	2.3458 (1.1668)	0.9417 (0.0430)	3.2620 (0.2206)	0.3392 (0.1229)	0.0255 (0.2742)	0.5857 0.7462	1.8964	0.0422	-273.41
M_M^T	0.9600 (0.0057)	4.3083 (4.0818)	0.9098 (0.0364)	0.1499 (0.0258)	0.3552 (0.3121)	-0.2234 (0.6669)	4.2563 0.1191	1.9832	0.0548	-262.76
M_A^E	0.9615 (0.0071)	3.4303 (1.5484)	0.9288 (0.0413)	0.3399 (0.1454)	0.3010 (0.1381)	-0.1052 (0.2938)	0.1486 0.9284	1.9771	0.9904	-270.95
M_M^E	0.9642 (0.0060)	5.7121 (6.9891)	0.8984 (0.0383)	0.0224 (0.0155)	0.2449 (0.5139)	-0.4293 (0.9053)	3.0585 0.2167	2.0246	0.9806	-262.83
				Panel B: $\gamma = 1$ imposed						
M_A^T	0.9639 (0.0074)	1	0.9358 (0.0465)	3.4270 (0.2383)	0.4254 (0.0777)	0.3120 (0.0928)	0.6341 0.7283	1.8346	0.0490	-273.94
M_M^T	0.9603 (0.0051)	1	0.9020 (0.0371)	0.1549 (0.0101)	0.5368 (0.0646)	0.3291 (0.0740)	7.8357 0.0199	1.8175	0.1998	-263.17
M_A^E	0.9642 (0.0072)	1	0.9360 (0.0444)	0.3236 (0.1632)	0.4366 (0.0740)	0.3340 (0.0798)	0.2912 0.8645	1.8577	0.9912	-272.74
M_M^E	0.9645 (0.0055)	1	0.8937 (0.0345)	0.0221 (0.0055)	0.5287 (0.0638)	0.3293 (0.0744)	8.7039 0.0129	1.7855	0.9860	-263.34

Notes: Figures in brackets are standard deviations based on the hessian of the loglikelihood function. The normality test is a Jarque–Bera joint skewness and kurtosis test, $\chi(2)$-distributed. The R^2–figures are based on a simple additive formulation, e.g. $p_t = p^E(d_t, c_t, z_t) + e_t$, where the only explanatory effect of z is through p^E. ℓ is the value of the loglikelihood function.

5.5. The Predictability Effect

The reduction in approximation errors caused by empirical rationality is clearly illustrated in Figure 2. The empirical price is shown together with the equilibrium price for both the theoretical and the empirical rationality model. Obviously, $\overset{\mathrm{T}}{p}$ explains very little of the empirical price variation. This *stock market excess volatility* is well documented by Shiller (1981) and Grossman and Shiller (1981).

The close empirical fit of $\overset{\mathrm{E}}{p}$ sharply contrast the failor of $\overset{\mathrm{T}}{p}$. The reason for this dramatic change is explained by the high autocorrelation of the approximation errors combined with the predictability effect. To simplify the analysis, ignore risk premiums and assume that future values of z and c are deterministic. Assuming additive specification and using (14) and (16), the model can be decomposed:

$$\begin{aligned} p_t &= \overset{\mathrm{E}}{p}(d_t, c_t, z_t) + z_t \\ &= \overset{\mathrm{T}}{p}(d_t, c_t) + \frac{\beta\zeta}{1-\beta\zeta} z_t + z_t \qquad (20) \\ &\approx \overset{\mathrm{T}}{p}(d_t, c_t) + 9z_t + z_t, \end{aligned}$$

when $\beta = 0.96$ and $\zeta = 0.94$. The last line of (20) shows, that when the empirical price differ from the one predicted by the theoretical model, the equilibrium price of the empirical model reacts very strongly. Thus, 10% of the original approximation errors now explain the empirical variation. The remaining 90% is accounted for by a change in the equilibrium price, due to the predictability of future approximation errors. This explains the drop in the estimated σ_z-values in Table 2 which is close to 90% for the additive specification.

With a fall in the σ_z estimate at 85%, the results for the multiplicative specification are simular to those of the additive specification.

5.6. Risk Premiums and Risk Effects

Empirically, the predictability effect far dominates the risk premium effect of approximation errors, but risk premiums in stock pricing models are interesting in their own right. Mehra and Prescott (1985) and the literature initiated by this study show that traditional rational expectation models are unable to explain the high empirical risk premium at stocks. This *equity premium puzzle* is particular troubling for Lucas (1978)-type models with endogenous price determination since the price volatility and the resulting risk premium generated by these models are very low even compared to Mehra and Prescott (1985)-type models. Empirical rationality and approximation errors are interesting in this respect, since a new type of uncertainty and a new type of risk premium is added to the model.

Table 3 shows the risk premiums and the risk effect implied by the four models. All numbers are calculated using the parameter estimates in Panel B of Table 2. In particular, they all share the same relative risk aversion factor, $\gamma = 1$. All figures have been calculated at the deterministic steady state and as the average of the figures implied by the realized empirical series. Since the effect of risk is not constant over the state space, a standard deviation is calculated to illustrate the size of this variation.

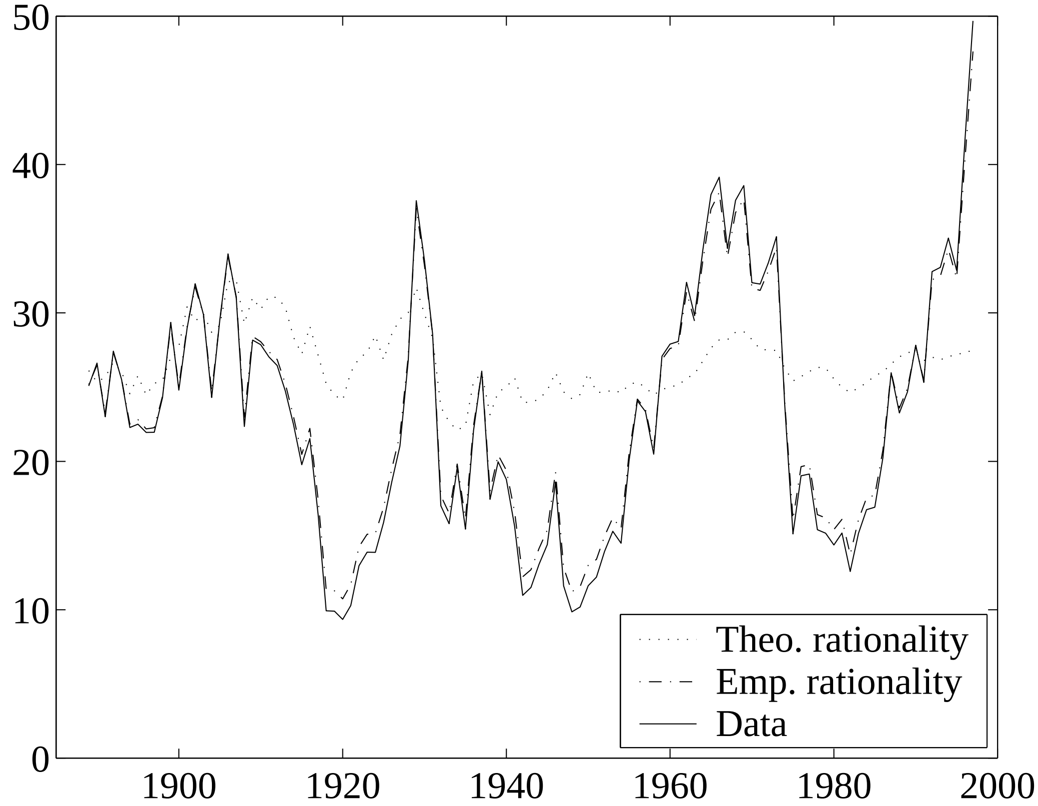


FIGURE 2: Empirical price and model prices.
The model prices are based at M_A^T and M_A^E and the relevant parameter values of Panel B of Table 2.

Starting with the dividend risk, the accompanying risk premium lowers the equilibrium price. The fall is approximately 0.5% for all models, but with a slightly larger fall for the additive specification. This should come as no surprise since the additive specification assign higher probability to the worst case scenarios. The sign of the price change is expected also. Since the shocks to consumption and dividends are positively correlated, dividends are a poor insurance against future consumption uncertainty.

Turning to the risk premiums attributed to the approximation errors, these are obviously only available for empirical rationality models. Again, positive correlation between the shocks to consumption and approximation errors causes the equilibrium price to fall. But now the size of the fall differs significantly across specifications. Even for the smallest fall, however, the effect of the approximation error risk is of the same magnitude as the traditional risk premium.

Finally, the risk premium of consumption uncertainty is rather constant over models, and raises the price of the stock with 0.3%. A price increase is expected since consumption uncertainty increases the demand for stocks which are useful for consumption smoothing.

TABLE 3

RISK PREMIUMS AS EQUILIBRIUM PRICE CHANGES IN %

		Risk premiums		
		Consumption	Dividends	Approximation errors
Model	Evaluation	π^c	π^d	π^z
M_A^T	Steady State	0.306	-0.500	-
	Empirical	0.306	-0.502	-
		(0.006)	(0.022)	
M_M^T	Steady State	0.313	-0.448	-
	Empirical	0.313	-0.448	-
		(0.003)	(0.004)	
M_A^E	Steady State	0.312	-0.508	-1.431
	Empirical	0.323	-0.643	-1.811
		(0.020)	(0.225)	(0.632)
M_M^E	Steady State	0.316	-0.440	-0.352
	Empirical	0.312	-0.434	-0.341
		(0.006)	(0.008)	(0.017)

Notes: *Steady state* is effects at the "deterministic steady state": $(d, c, z) = (0, 0, 0)/(1, 1, 1)$ for the additive/multiplicative specification. *Empirical* is average effects implied by the models for the empirical data-series, 1889-1997. Figures in brackets are standard deviations. All figures are based on the estimates in Panel B, Table 2.

5.7. EXPLOIDING EXCESS RETURN OPPORTUNITIES

The descriptive success of $\overset{\mathrm{T}}{p}$ is limited compared to $\overset{\mathrm{E}}{p}$. This subsection compares the normative properties of theoretical and empirical rationality.

Due to the strong theoretical foundation of the original theoretical model, observed price deviations from $\overset{\mathrm{T}}{p}$ might offer excess return opportunities to the investor. Therefore, assume that the theoretical model in (4) and (5.A) is a precise description of the rational investors micro-situation and assume that the investor uses the model for price predictions, that is theoretical expectations. Consider the utility gained by giving up consumption worth one unit of utility at time $t-1$, investing the funds in stocks, and consuming the total value of the investment at time t. The change in the overall utility gained by this transaction is denoted ∂u_t. According to the Euler equation, the expected utility gain should be zero in equilibrium. However, that is not the case if observed prices differ from the equilibrium price.

Table 4 shows a few statistics on the relation between the actual realized utility gains ∂u_t and the one-year-in-advance expected utility gains ∂u_t^e based on the theoretical rationality model M_A^T. There is a significant linear dependency between ∂u_t and ∂u_t^e and the correlation is almost 0.25. The theoretical model is therefore

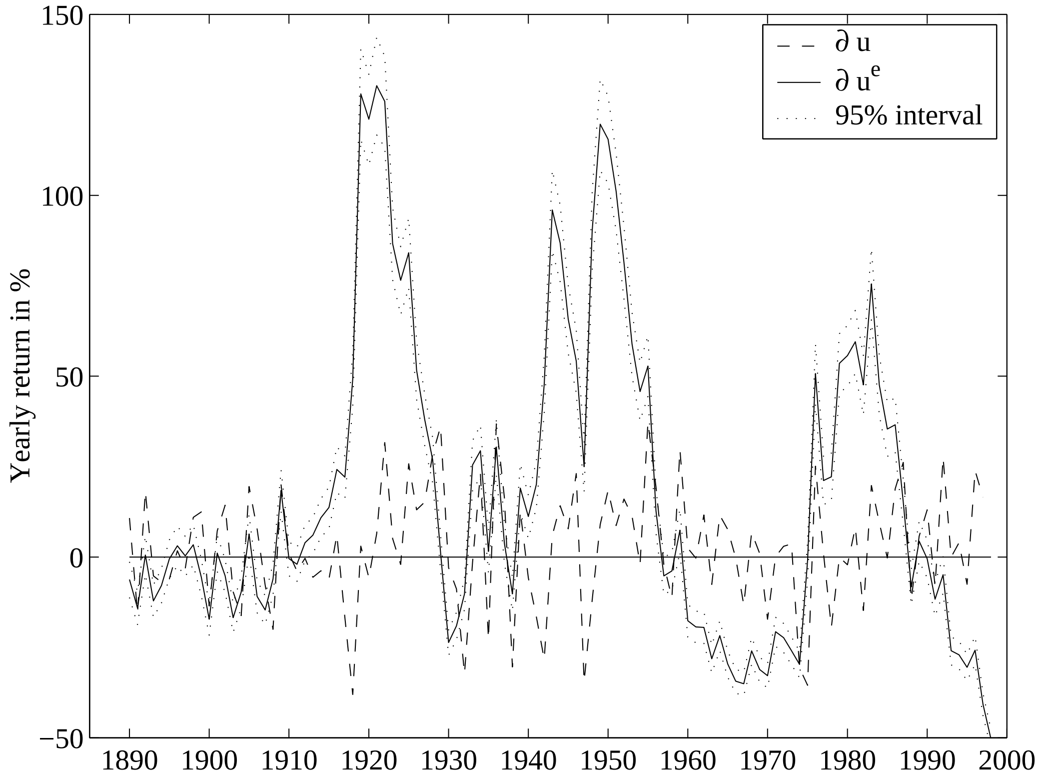


FIGURE 3: Utility gains with theoretical expectations.
The expectations are based at theoretical a theoretical model, M_A^T. The confidence interval is calculated using quasi monte carlo integration, (1000 Niederreider points).

capable of predicting utility gain opportunities, but with a regression coefficient at 0.1, the model is overestimating the size of the utility gain opportunities with a factor 10.

Figure 3 shows ∂u_t and ∂u_t^e together with the 95% one-year-in-advance confidence interval for ∂u_t^e based on M_A^T. Obviously, the confidence with which the theoretical model predicts ∂u_t is far to optimistic. Only 17% of the realized ∂u_t-values falls inside the 95% confidence interval. Although the investment strategy implied by the model will earn a positive utility gain in the long run, Figure 3 shows that there are at least 3 periods in the sample that would have ruined the investors that took the full consequence of the model's predictions. In the years around 1920, 1950, and 1980 the model would have suggested to borrow, if possible, at a fixed interest rate of up to 50% and investing the funds in the S&P 500 index. Therefore, the investment strategy suggested by the model seems to be far to aggressive.[11]

Now consider a simular experiment with empirical rationality imposed. Assume

[11] It should be noticed that since the model is a general equilibrium model it does not allow the investor to deviate from $w = 0$. Thus, strictly speaking, only marginal changes can be analyzed.

TABLE 4

EXCESS RETURN STATISTICS

Model	$\partial u_t = \alpha\ \partial u_t^e + \epsilon_t$ α	R^2	DW	$Std(\partial u_t)$	$Std(\partial u_t^e)$	$Corr$
M_A^T	0.100 (0.033)	0.058	1.842	16.33	42.44	0.2436
M_A^E	0.914 (0.310)	0.054	1.830	16.33	4.713	0.2426

Notes: α is the least square regression coefficient. DW is the Durbin-Watson test statistic.$Std(\partial u_t)$ and $Std(\partial u_t^e)$ are standard deviations for actual and expected utility return. "Corr" is the correlation between ∂u_t and ∂u_t^e.

that an investor believe that his "micro-situation" is well described by (4) and (5.A). However, the investor realize that at the "macro-level" the price will be subject to approximation errors as described by M_A^E. Therefore, the investor forms empirical expectations.

The introduction of empirical expectations improves the normative properties significantly. The relevant α-estimate in Table 4 is not significantly different from 1. Thus, the size of ∂u_t is correctly predicted on average by M_A^E. Figure 4, which should be compared with Figure 3, shows ∂u_t as well as ∂u_t^e and the 95% confidence interval based on the empirical rationality model M_A^E. With 96.3% of the predictions inside the 95% significance band, the risk of investing is accurately predicted by the model.

Figure 4 might explain why serious mispricing is not traded away by speculators. The expected gain might be positive but the risk involved is considerable. Especially, the risk seems to increase in the years around 1920, 1950, and 1980 when investments in the S&P500 are predicted to be most profitable. Therefore, investment strategies that recognize approximation errors should be much more defensive.

6. CONCLUSION

This paper has questioned the usual working hypothesis that considers structural economic models to be true. Besides the obvious lack of intuitive justification, this hypothesis causes practical problems to the researcher in form of statistical degeneracy. Moreover, the rational agents considered in the models are not allowed to take model uncertainty optimally into account, even in situations where the predictions of the model are very poor.

These problems were dealt with using an approximation error approach to model uncertainty. Both the researcher and the agents abandon the usual working hypothesis and recognize that the implications of the model will be subject to approximation errors. Subjective believes with respect to the approximation errors were

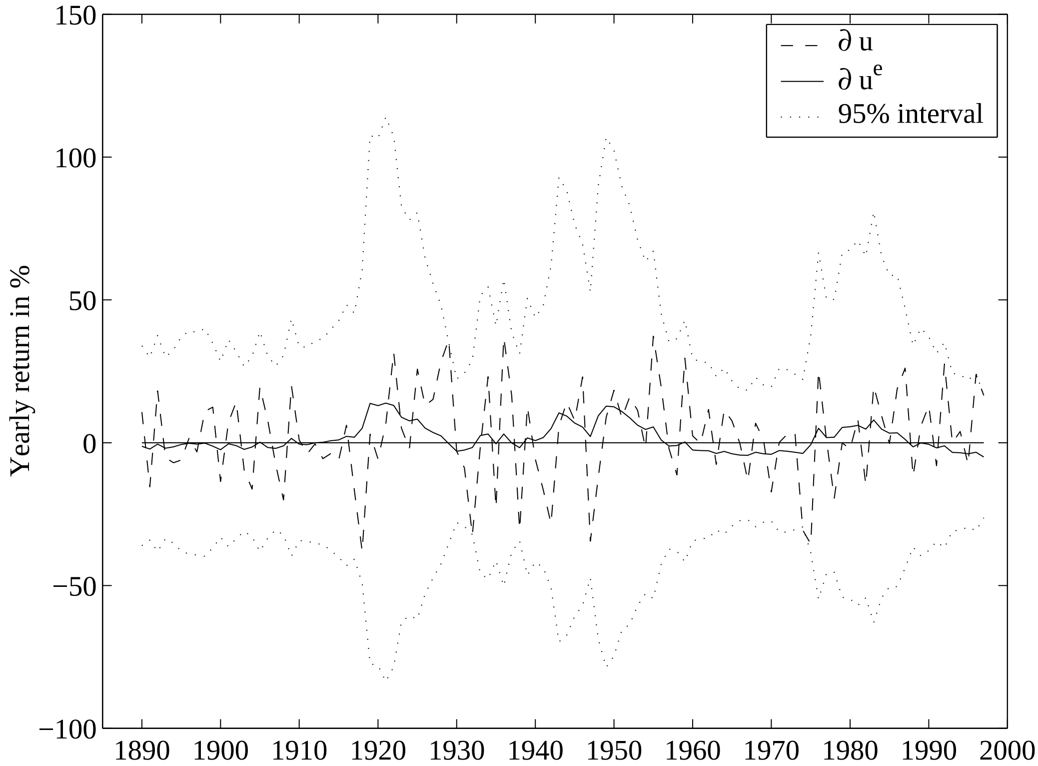


FIGURE 4: Utility gains with empirical expectations
The expectations are based at theoretical a theoretical model, M_A^E. The confidence interval is calculated using quasi monte carlo integration, (1000 Niederreider points).

formed in order to break the statistical degeneracy and to allow model uncertainty to influence the optimal behavior of the agents.

The importance of the approach was illustrated using a Lucas (1978)-type asset pricing model. The implications of the model with respect to U.S. stock prices were addressed. The significant difference between the empirical prices and the equilibrium price was accounted for by approximation errors. As a consequence, the investors re-optimized their investment decisions with a new and rather different equilibrium price as a result. The new equilibrium price behavior had both descriptive and normative value.

From a descriptive perspective, two important stock marked puzzles were addressed. The *stock marked excess volatility* documented by Shiller (1981) and Grossman and Shiller (1981) was significantly reduced. Due to the sensitivity of the equilibrium price with respect to observed approximation errors, the distance between the empirical prices and the equilibrium prices was reduced with 90%.

Also the *equity premium puzzle* documented by Mehra and Prescott (1985) was addressed. Approximation errors represent uncertainty about the appropriate asset pricing model, and this uncertainty requires additional risk premiums. The equity premium puzzle was not solved, but the analysis showed that the risk premiums generated was doubbled due to the approximation errors.

The normative properties were also improved. If the empirical shortcomings of the model are ignored, the model will predict excess return opportunities far greater than the once actually realized. Moreover, the model will significantly underestimate the risk involved in exploiting these opportunities. When approximation errors are taken into account, the size of the predictions as well as the estimate of the uncertainty involved are unbiased. Thus, the model provides a better advices with respect to stock investments.

Summing up, the normative results suggest that investors should take the empirical shortcomings of their structural asset pricing models into account. The descriptive results suggest that the investors in fact do that. If this is true, researches might consider approximation errors for two reasons. First, the descriptive and normative relevance of their models might improve. Secondly, the symmetry between the researchers and the empirical agents believes about theoretical models might be reestablished.

Appendix A Results for the Multiplicative Specification

This appendix considers a number of analytical results for the asset pricing model with the multiplicative in (5.M) and (17).

A.1 Closed Form Solutions, the Empirical Model

Consider equation (14) in section 4:

$$\overset{\mathrm{E}}{p}(d_0, c_0, z_0) = E_0 \sum_{t=1}^{\infty} \beta^t \frac{u'(c_t)}{u'(c_0)} d_t z_{t-1} z_{t-2} \dots z_1. \tag{21}$$

Since

$$\begin{aligned} d_t &= d_0^{\delta^t} \exp(\epsilon_t^d + \delta\epsilon_{t-1}^d + \delta^2\epsilon_{t-2}^d + \dots + \delta^{t-1}\epsilon_1^d), \\ c_t &= c_0^{\theta^t} \exp(\epsilon_t^c + \theta\epsilon_{t-1}^c + \theta^2\epsilon_{t-2}^c + \dots + \theta^{t-1}\epsilon_1^c), \\ z_t &= z_0^{\zeta^t} \exp(\epsilon_t^z + \zeta\epsilon_{t-1}^z + \zeta^2\epsilon_{t-2}^z + \dots + \zeta^{t-1}\epsilon_1^z), \end{aligned}$$

and

$$\begin{aligned} &z_t z_{t-1} \dots z_1 \\ &\quad = z_0^{\zeta^t + \zeta^{t-1} + \dots + \zeta} \exp\left(\epsilon_t^z + (1+\zeta)\epsilon_{t-1}^z + \dots + \left(1 + \zeta + \dots + \zeta^{t-1}\right)\epsilon_1^z\right) \\ &\quad = z_0^{\frac{\zeta - \zeta^{t+1}}{1-\zeta}} \exp\left(\tfrac{1-\zeta}{1-\zeta}\epsilon_t^z + \tfrac{1-\zeta^2}{1-\zeta}\epsilon_{t-1}^z + \dots + \tfrac{1-\zeta^t}{1-\zeta}\epsilon_1^z\right), \end{aligned}$$

equation (21) can be rewritten as

$$\overset{\mathrm{E}}{p}(d_0, c_0, z_0) = \sum_{t=1}^{\infty} \beta^t c_0^{-\gamma(\theta^t-1)} d_0^{\delta^t} z_0^{\frac{\zeta-\zeta^t}{1-\zeta}} E_0 \Big[\exp \Big(\epsilon_t^d + \delta\epsilon_{t-1}^d + \cdots + \delta^{t-1}\epsilon_1^d$$

$$-\gamma(\epsilon_t^c + \theta\epsilon_{t-1}^c + \cdots + \theta^{t-1}\epsilon_1^c)$$

$$+ \tfrac{1-\zeta^0}{1-\zeta}\epsilon_t^z + \tfrac{1-\zeta^1}{1-\zeta}\epsilon_{t-1}^z + \cdots + \tfrac{1-\zeta^{t-1}}{1-\zeta^0}\epsilon_1^z \Big) \Big]$$

$$= \sum_{t=1}^{\infty} \beta^t c_0^{-\gamma(\theta^t-1)} d_0^{\delta^t} z_0^{\frac{\zeta-\zeta^t}{1-\zeta}} E_0 \exp \Big(f_0^\top \epsilon_t + f_1^\top \epsilon_{t-1} + \cdots + f_{t-1}^\top \epsilon_1 \Big)$$

where

$$f_s = \begin{bmatrix} \delta^s \\ -\gamma\theta^s \\ \frac{1-\zeta^s}{1-\zeta} \end{bmatrix} \quad \text{and} \quad \epsilon_s = \begin{bmatrix} \epsilon_s^d \\ \epsilon_s^c \\ \epsilon_s^z \end{bmatrix}.$$

Calculating the expected value of the lognormal distributed variables gives

$$\overset{\mathrm{E}}{p}(d_0, c_0, z_0) = \sum_{t=1}^{\infty} \beta^t c_0^{-\gamma(\theta^t-1)} d_0^{\delta^t} z_0^{\frac{\zeta-\zeta^t}{1-\zeta}} \times \exp \Big(\mu^\top f_0 + \tfrac{1}{2} f_0^\top \Omega f_0$$

$$+\mu^\top f_1 + \tfrac{1}{2} f_1^\top \Omega f_1$$

$$\vdots$$

$$+\mu^\top f_{t-1} + \tfrac{1}{2} f_{t-1}^\top \Omega f_{t-1}$$

$$= \sum_{t=1}^{\infty} \beta^t c_0^{-\gamma(\theta^t-1)} d_0^{\delta^t} z_0^{\frac{\zeta-\zeta^t}{1-\zeta}} \Psi_t$$

where

$$\Psi_t = \exp\left(\sum_{s=0}^{t-1} h_s \right) \quad \text{and} \quad h_s = \mu^\top f_s + \tfrac{1}{2} f_s^\top \Omega f_s$$

A.2 Closed Form solutions, the Theoretical Model

The theoretical model is a special case of the empirical model where $z_t = 1$ and z_t is deterministic for $t > 0$:

$$\overset{\mathrm{E}}{p}(d_0, c_0, z_0) = \sum_{t=1}^{\infty} \beta^t c_0^{-\gamma(\theta^t-1)} d_0^{\delta^t} \Lambda_t$$

where Λ_t, is defined like Ψ_t except that

$$\overset{\mathrm{T}}{f}_s = \begin{bmatrix} \delta^s \\ -\gamma\theta^s \\ 0 \end{bmatrix}$$

replaces f_s.

A.3 Determination of μ

Consider first the dividend process. For given δ and σ_d^2, μ_d should be determined such that the unconditional expectation of the dividends is one:

$$E[d_t] = 1.$$

Using the transition equation for dividends for repeated substitution gives

$$\begin{aligned} d_t &= d_{t-1}^{\delta} \exp \epsilon_t^d \\ &= \left(d_{t-2}^{\delta} \exp \epsilon_{t-1}^d\right)^{\delta} \exp \epsilon_t^d \\ &= d_{t-2}^{\delta^2} \exp(\epsilon_t^d + \delta \epsilon_{t-1}^d) \\ &= d_{t-\tau}^{\delta^\tau} \exp(\epsilon_t^d + \delta \epsilon_{t-1}^d + \delta^2 \epsilon_{t-2}^d + \cdots + \delta^{\tau-1} \epsilon_{t-\tau+1}^d) \\ &\to \exp(\epsilon_t^d + \delta \epsilon_{t-1}^d + \delta^2 \epsilon_{t-2}^d + \cdots) \text{ for } \tau \to \infty. \end{aligned}$$

Hence,

$$\begin{aligned} E[d_t] &= \exp\left((1 + \delta + \delta^2 + \cdots)\mu_d + \tfrac{1}{2}(1 + \delta^2 + \delta^4 + \cdots)\sigma_d^2\right) \\ &= \exp\left(\tfrac{1}{1-\delta}\mu_d + \tfrac{1}{2(1-\delta^2)}\sigma_d^2\right) \end{aligned}$$

and

$$\mu_d = -\tfrac{1-\delta}{2(1-\delta^2)}\sigma_d^2 \Rightarrow E[d_t] = 1.$$

Using the same arguments for the consumption process gives,

$$\mu_c = -\tfrac{1-\theta}{2(1-\theta^2)}\sigma_c^2 \Rightarrow E[c_t] = 1.$$

When determining μ_z, it should be taken in consideration that $f_i^\top \to [0\ 0\ \frac{1}{1-\zeta}]$ for $i \to \infty$. This causes

$$h_s \to \tfrac{1}{1-\zeta}\mu_z + \tfrac{1}{2(1-\zeta)^2}\sigma_z^2 \quad \text{for} \quad s \to \infty$$

and if the limit value of h_s is positive will

$$\Psi_t = \exp\left(\sum_{s=0}^{t-1} h_s\right)$$

increase exponentially in the limit. Hence, the equilibrium price will be infinite. To ensure a finite the mean of ϵ_z must be bounded,

$$\mu_z \leq \tfrac{-\sigma_z^2}{2(1-\zeta)}.$$

Therefor, $\mu_z = \frac{-\sigma_z^2}{2(1-\zeta)}$ is assumed in the analysis.

A.4 Definition of Risk Premiums and their Closed Form Expressions

The risk premiums are defined as

$$\begin{aligned}\pi_0^z &= E_0\left[\sum_{t=1}^{\infty}\widetilde{\beta}^t E_0(d_t)\left(z_{t-1}z_{t-2}\cdots z_1 - E_0(z_{t-1}z_{t-2}\cdots z_1)\right)\right]\\ &= \overset{\mathrm{E}}{p}_0(\overline{d}) - \overset{\mathrm{E}}{p}_0(\overline{d},\overline{z}),\end{aligned}$$

$$\begin{aligned}\pi_0^d &= E_0\left[\sum_{t=1}^{\infty}\widetilde{\beta}^t (d_t - E_0(d_t))E_0(z_{t-1}z_{t-2}\cdots z_1)\right]\\ &= \overset{\mathrm{E}}{p}_0(\overline{z}) - \overset{\mathrm{E}}{p}_0(\overline{d},\overline{z}),\\ \pi_0^c &= E_0\left[\sum_{t=1}^{\infty}\beta^t \frac{u'(c_t)-u'(\overline{c}_t)}{u'(c_0)}E_0(d_t)E_0(z_{t-1}z_{t-2}\cdots z_1)\right]\\ &= \overset{\mathrm{E}}{p}_0(\overline{d},\overline{z}) - \overset{\mathrm{E}}{p}_0(\overline{d},\overline{c},\overline{z}).\end{aligned}$$

where:

$$\begin{aligned}\overset{\mathrm{E}}{p}_0(\overline{z}) &= E_0\left[\sum_{t=1}^{\infty}\widetilde{\beta}^t d_t E_0(z_{t-1}z_{t-2}\cdots z_1)\right],\\ \overset{\mathrm{E}}{p}_0(\overline{d}) &= E_0\left[\sum_{t=1}^{\infty}\widetilde{\beta}^t E_0(d_t) z_{t-1}z_{t-2}\cdots z_1\right],\\ \overset{\mathrm{E}}{p}_0(\overline{d},\overline{z}) &= E_0\left[\sum_{t=1}^{\infty}\widetilde{\beta}^t E_0(d_t)E_0(z_{t-1}z_{t-2}\cdots z_1)\right],\\ \overset{\mathrm{E}}{p}_0(\overline{d},\overline{c},\overline{z}) &= E_0\left[\sum_{t=1}^{\infty}\beta^t \frac{u'(\overline{c}_t)}{u'(c_0)}E_0(d_t)E_0(z_{t-1}z_{t-2}\cdots z_1)\right].\end{aligned}\tag{22}$$

Since all the artificial prices in (22) have closed form solutions, closed form solutions are also available for the risk premiums. Consider first $\overset{\mathrm{E}}{p}(\overline{z})$:

$$\begin{aligned}\overset{\mathrm{E}}{p}_0(\overline{z}) = &\sum_{t=1}^{\infty}\beta^t c_0^{-\gamma(\theta^t-1)} d_0^{\delta^t} z_0^{\frac{\zeta-\zeta^t}{1-\zeta}}\\ &\times E_0\left[\exp\left(\epsilon_t^d + \delta\epsilon_{t-1}^d + \cdots + \delta^{t-1}\epsilon_1^d - \gamma(\epsilon_t^c + \theta\epsilon_{t-1}^c + \cdots + \theta^{t-1}\epsilon_1^c)\right)\right]\\ &\times E_0\left[\exp\left(\tfrac{1-\zeta^0}{1-\zeta}\epsilon_t^z + \tfrac{1-\zeta^1}{1-\zeta}\epsilon_{t-1}^z + \cdots + \tfrac{1-\zeta^{t-1}}{1-\zeta^0}\epsilon_1^z\right)\right]\\ = &\sum_{t=1}^{\infty}\beta^t c_0^{-\gamma(\theta^t-1)} d_0^{\delta^t} z_0^{\frac{\zeta-\zeta^t}{1-\zeta}}\exp\left(\sum_{s=0}^{t-1}\mu^\top f_s + \tfrac{1}{2}f_s^\top\Omega^{\overline{z}}f_s\right)\end{aligned}$$

which is equal to the closed form solution to $\overset{\mathrm{E}}{p}_0$ except that

$$\Omega^{\overline{z}} = \begin{bmatrix}\sigma_d^2 & \sigma_{d,c} & 0\\ \sigma_{d,c} & \sigma_c^2 & 0\\ 0 & 0 & \sigma_z^2\end{bmatrix}$$

replaces Ω. Likevise, the closed form solutions for $\overset{\mathrm{E}}{p}_0(\overline{d})$ and $\overset{\mathrm{E}}{p}_0(\overline{d},\overline{z})$ equals the solution to $\overset{\mathrm{E}}{p}_0$, except that Ω is replaced with

$$\Omega^{\overline{d}} = \begin{bmatrix} \sigma_d^2 & 0 & 0 \\ 0 & \sigma_c^2 & \sigma_{d,c} \\ 0 & \sigma_{d,c} & \sigma_z^2 \end{bmatrix} \quad \textit{and} \quad \Omega^{\overline{d},\overline{z}} = \begin{bmatrix} \sigma_d^2 & 0 & 0 \\ 0 & \sigma_c^2 & 0 \\ 0 & 0 & \sigma_z^2 \end{bmatrix}.$$

The case of $\overset{\mathrm{E}}{p}_0(\overline{d},\overline{c},\overline{z})$ is a bit more complicated due to the non-linearity of the utility function:

$$\begin{aligned} \overset{\mathrm{E}}{p}_0(\overline{z},\overline{c},\overline{d}) = & \sum_{t=1}^{\infty} \beta^t c_0^{-\gamma(\theta^t-1)} d_0^{\delta^t} z_0^{\frac{\zeta-\zeta^t}{1-\zeta}} \\ & \times E_0 \exp(\epsilon_t^d + \delta\epsilon_{t-1}^d + \cdots + \delta^{t-1}\epsilon_1^d) \\ & \times E_0 \exp\left(\tfrac{1-\zeta^0}{1-\zeta}\epsilon_t^z + \tfrac{1-\zeta^1}{1-\zeta}\epsilon_{t-1}^z + \cdots + \tfrac{1-\zeta^{t-1}}{1-\zeta^0}\epsilon_1^z\right) \\ & \times \left(E_0 \exp(\epsilon_t^c + \theta\epsilon_{t-1}^c + \cdots + \theta^{t-1}\epsilon_1^c)\right)^{-\gamma}. \end{aligned}$$

Since

$$\begin{aligned} & \left(E_0 \exp\left(\epsilon_t^c + \theta\epsilon_{t-1}^c + \cdots + \theta^{t-1}\epsilon_1^c\right)\right)^{-\gamma} \\ & = \exp\left(-\gamma\mu_c - \gamma\tfrac{1}{2}\sigma_c^2 - \gamma\theta\mu_c - \gamma\tfrac{1}{2}\theta^2\sigma_c^2 - \ldots - \gamma\theta^{t-1}\mu_c - \gamma\tfrac{1}{2}\theta^{2(t-1)}\sigma_c^2\right) \end{aligned}$$

the closed form solution to $\overset{\mathrm{E}}{p}_0(\overline{d},\overline{c},\overline{z})$ equals the solution to $\overset{\mathrm{E}}{p}_0$ exept that

$$\Omega^{\overline{d},\overline{z},\overline{c}} = \begin{bmatrix} \sigma_d^2 & 0 & 0 \\ 0 & -\frac{\sigma_c^2}{\gamma} & 0 \\ 0 & 0 & \sigma_z^2 \end{bmatrix}.$$

replaces Ω.

Appendix B Numerical Solution Method

The numerical solutions of the models with additive specifications are projection method solutions to the Euler-equations, see Judd (1992).

The approximation basis consist of tensor cubic b-splines, see de Boor (1978). For the theoretical and the empirical model, 5×5 and $5 \times 5 \times 5$ b-spline elements were used. Integration with respect to expectations where calculated using 4×4 and $2 \times 2 \times 2$ Hermite points. The size of the state space was chosen such that all integration points were interior. The b-spline coefficients where chosen to minimize the squared numerical approximation errors at 13×13 and $13 \times 13 \times 13$ approximation points equally spaced over the state space.

With respect to precision, the Euler equation seemed to be violated by a maximum of 1E-4% over the state space. This precision is, of course, conditioned on the precision of the integration. However, the quasi Monte Carlo methods used for calculating the confidence intervals in figure 3 and 4 also showed that the Hermite integration was quite accurate despite the low number of points.

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