WP 2004-4

Upper Bonds on Numerical Approximation Errors

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ISBN 87-90705-80-7
ISSN 0903-0352

UPPER BOUNDS ON NUMERICAL APPROXIMATION ERRORS

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September 9, 2004

This paper suggests a method for determining rigorous upper bounds on approximation errors of numerical solutions to infinite horizon dynamic programming models. Bounds are provided for approximations of the value function and the policy function as well as the derivatives of the value function. The bounds apply to more general problems than existing bounding methods do. For instance, since strict concavity is not required, linear models and piecewise linear approximations can be dealt with. Despite the generality, the bounds perform well in comparison with existing methods even when applied to approximations of a standard (strictly concave) growth model.

KEYWORDS: Numerical approximation errors, Bellman contractions, Error bounds

1. INTRODUCTION

A key element in economics is the Ramsey (1928) model and variants hereof. Unfortunately, within these models closed form solutions are often not available and numerical approximations are often the best available alternative. A natural question to ask is how such approximations should be calculated, especially the class of discrete time growth models which have been subject to research in this respect. For instance, Taylor and Uhlig (1990) and ten companion papers were dedicated to the study of precision and speed regarding a number of numerical methods applied to a standard growth model. A similar study for a model with occasionally binding constraints can be found in Christiano and Fisher (2000). Such studies provide valuable information when the model of interest is similar to the model studied. However, as noted by Taylor and Uhlig (1990), one should generally not trust the results blindly. Instead, accuracy checks for the particular problem considered should be calculated.

A number of accuracy checks have been suggested to test the precision of numerical approximations of the class of models subject to the above studies. Both den Haan and Marcet (1994) and Judd (1992) suggest measuring the precision of a given numerical approximation by the size of Euler equation violations. Santos (2000) is able to establish

*I would like to thank Henning Bunzel, Bent Jesper Christensen, Kenneth Judd, Morten O. Ravn, Martin Richter, and Harald Uhlig for valuable comments and suggestions.

a rigorous relationship from the size of such Euler equation violations to the precision of the value function implied by the approximate policy function, (Lemma 3.1) by assuming strong concavity, interior solutions, and "nondivergent orbits". Using this relationship, the precision of the approximate policy function can be bounded in the sup-norm, (Lemma 3.2). As noted by Santos (2000), the nondivergent orbits- assumption limits the application of the results. Important asymptotic results are nevertheless obtained.[1]

Interior solution and nondivergent orbit assumptions are only used by Santos (2000) in the first step to bound the precision of the implied value function. One way around these assumptions is to focus initially on numerical approximations of the value function and use the Bellman equation violations and the well-known contraction mapping result to bound the value function approximation errors. If precision measures are needed for the policy function approximation, Santos (2000, Lemma 3.2), which requires only strong concavity, might be used, see Maldonado and Svaiter (2001).

This paper suggests an alternative procedure for calculating numerical error bounds for policy function approximations. Since concavity is not required to be strong, common features like linear utility, constant returns to scale, and piecewise linear approximations are allowed. Despite being more general, the suggested bounds often turn out to be more precise than existing bounding procedures, even for a strictly concave standard test problem.

The paper is organized as follows. Section 2 describes the setup. Section 3 argues that the contraction mapping property of the Bellman equation is useful for numerical error bounding for a wider set of problems than usually recognized by the literature. Section 4 shows how the contraction mapping error bound provides bounds on the first order derivatives of the true value function, which again impose bounds the the optimal policy. Section 5 investigates the properties of the various bounding procedures for two test models. In the first example, only the bounding procedure suggested below applies. The second example is a the standard growth model. Section 6 concludes.

2. THE GENERAL SETUP

The popular setup of Stokey and Lucas (1989, Chapter 9.2, Assumptions 9.4-9.7) is adapted below:

ASSUMPTION 1: Let $(X, \mathcal{X})$ be the space of possible values for the endogenous state variables, where X is a convex set in $\mathbb{R}^l$ and $\mathcal{X}$ is its Borel subsets.

ASSUMPTION 2: Let $(Z, \mathcal{Z})$ be the space of possible values for the exogenous state variables, where Z is either a countable set and $\mathcal{Z}$ is the σ-algebra containing all subsets of $\mathcal{Z}$; or a compact (Borel) set in $\mathbb{R}^k$ with its Borel subsets $\mathcal{Z}$. Let $Q : (Z, \mathcal{Z}) \to [0, 1]$ be the Markov transition function for the exogenous state variables, where Q has the Feller property. Let $(Z^t, \mathcal{Z}^t) = (Z \times \ldots \times Z, \mathcal{Z} \times \ldots \times \mathcal{Z})$ be a t-fold product space of partial histories $z^t = (z_1, \ldots, z_t), t = 1, 2, \ldots$, and let $\mu^t(z_0, \cdot) : \mathcal{Z}^t \to [0, 1]$ be a relevant probability measure for every $z_0 \in Z$.

[1] Accuracy checks are also provided by Santos and Vigo-Aguiar (1998), but these are valid for a specific numerical approximation method only.

ASSUMPTION 3: Let $\Gamma : X \times Z \to X$ be a nonempty, compact-valued, and continuous correspondence describing the feasible choices of next periods endogenous state variables.

ASSUMPTION 4: Let $F : A \to \mathbb{R}$ be a bounded and continuous one-period return function, where $A = \{(x, y, z) \in X \times X \times Z : y \in \Gamma(x, z)\}$ is the graph of Γ. Let $\beta \in (0, 1)$ be the discount factor.

The problem is finding a measurable policy $\pi : \pi_0 \in X$ and $\pi_t : Z^t \to X$, $t = 1, 2, \ldots$ that solves

$$\begin{aligned} &\sup_{\pi} \sum_{t=0}^{\infty} \int_{Z^t} F(\pi_{t-1}, \pi_t, z_t) \mu^t(z_0, dz^t) \equiv V(x_0, z_0) \\ &\text{subject to} \\ &\pi_t \in \Gamma(\pi_{t-1}, z_t), \quad (x_0, z_0) = (\pi_{-1}, z_0) \text{ fixed.} \end{aligned} \tag{1}$$

To solve the problem in (1), it is normal to turn to the Bellman equation. Consider the operator $T : C(X \times Z) \to C(X \times Z)$ given by

$$(TW)(x, z) = \sup_{y \in \Gamma(x,z)} \left\{ F(x, y, z) + \beta \int W(y, z'), Q(z, dz') \right\} \tag{2}$$

where $W \in C(X \times Z)$ and C denote the space of continuous functions. It can be showed that T is a contraction and that if a measurable fixed point exists, it is unique and equal to V as defined in (1):

$$\|T^n W - V\| \leq \beta^n \|W - V\|, \quad n = 1, 2, \ldots \tag{3}$$

for any $W \in C(X \times Z)$ where $\| \cdot \|$ denote the sup-norm. Moreover, the correspondence $G : X \times Z \to X$ defined by

$$G(x, z) = \operatorname*{argsup}_{y \in \Gamma(x,z)} \left\{ F(x, y, z) + \beta \int V(y, z') Q(z, z') \right\} \tag{4}$$

generates optimal policies according to (1).

3. NUMERICAL ERROR BOUNDS FOR THE VALUE FUNCTION

The contraction property of T is useful from at least two points of view. First, since $T^n W \to V$ for $n \to \infty$, the contraction suggests the dynamic programming procedure: Use W and (2) to calculate TW, then use TW and (2) to calculate $T^2 W$, etc. until $T^n W$ is supposed to be "close" to V. Second, since

$$\|W - V\| \leq \frac{1}{1 - \beta} \|TW - W\| \tag{5}$$

follows from (3), the distance between any function $W \in C(X \times Z)$ and the true, but unknown, value function V can be bound if the distance between TW and W can be calculated.[2]

In numerical analysis, both the dynamic programming procedure and the error bounding procedure are affected by numerical errors, and some of these errors are the same. This might easily lead to the belief that if numerical errors destroy the convergence properties of the dynamic programming procedure, they will also destroy the validity of the error bounding procedure, e.g. Santos (2000, p. 1378). This, however, is not true in general.

The main reason for the failure of the dynamic programming procedure when applied to continuous state spaces is *representation errors*. In order to determine T^2W, a continuous representation of TW must be stored in the computer. Continuous functions without closed form expressions cannot be stored perfectly, and representation errors will arise. The effect of these errors can be significant and hard to bound in practice. Therefore, convergence of T^nW in a numerical setting cannot be guaranteed except for special cases, e.g. Santos and Vigo-Aguiar (1998).

Representation errors have little to do with the numerical properties of the error bound in (5), however. Because TW is not represented, no representation errors are involved. The numerical errors involved in calculating the right-hand side of (5) can be separated into three types: *integration errors*, *optimization errors*, and *norm errors*. These errors are usually of less importance than representation errors. In fact, they are often neglected by numerical precision measures in the literature.

Integration errors ϵ^I arise when the exact integration in (2) is substituted with numerical integration, $\widehat{\int}$. Define

$$\epsilon^I(x,y,z) = \int W(y,z')Q(z,dz') - \widehat{\int} W(y,z')Q(z,dz')$$

where the *hat* both here and later denotes operators and quantities subject to numerical errors. The optimization errors arise when exact optimization in (4) is replaced by a numerical optimization procedure, $\widehat{\operatorname{argsup}}$. Let the set of optimal policies calculated be given by:

$$\hat{G}(x,z) = \underset{y\in\Gamma(x,z)}{\widehat{\operatorname{argsup}}} \left\{ F(x,y,z) + \widehat{\int} W(y,z')Q(z,dz') \right\}. \tag{6}$$

The numerical version of $Tf(x,z)$ is then:

$$\widehat{T}W(x,z) = F(x,\hat{g},z) + \beta \widehat{\int} W(\hat{g},z')Q(z,dz'), \quad \hat{g} \in \hat{G}.$$

Viewed separately, integration and optimization errors are standard numerical error terms. In this case, however, they are not independent. The integration error depends on the optimal policy calculated and is affected by the choice of optimum $\widehat{g}$ of the optimization algorithm. Therefore, the optimization algorithm might not determine the true policy of (4), even if the optimization algorithm is perfect. To circumvent this dependence,

[2] It is not always appreciated that W can be an arbitrary continuous function. In particular, W does not need to be the result of a specific solution algorithm such as the contraction mapping algorithm.

define the optimization error with respect to the optimal policy of (6) instead of the true optimal policy in (4):

$$\epsilon^M(x,z) = \sup_{y\in\Gamma(x,z)} \left\{ F(x,y,z) + \beta \widehat{\int} W(y,z'), Q(z,dz') \right\} - \widehat{T}W(x,z).$$

The errors can then be separated into pure optimization error and integration error terms:

$$\begin{aligned} & TW(x,z) \\ & \quad = \sup_{y\in\Gamma(x,z)} \left\{ F(x,y,z) + \beta \widehat{\int} W(y,z')Q(z,dz') + \epsilon^I(x,y,z) \right\} \\ & \quad \leq \sup_{y\in\Gamma(x,z)} \left\{ F(x,y,z) + \beta \widehat{\int} W(y,z')Q(z,dz') \right\} + \sup_{y\in\Gamma(x,z)} \{\epsilon^I(x,y,z)\} \\ & \quad = \widehat{T}W(x,z) + \epsilon^M(x,z) + \sup_{y\in\Gamma(x,z)} \{\epsilon^I(x,y,z)\}. \end{aligned}$$

Thus, TW can be calculated for any $(x,z) \in X \times Z$ with a precision given by two error terms for which error bounds from standard numerical analysis are often available.

Finally, let $\hat{\|} \cdot \hat{\|}$ denote the calculated norm and let ϵ^N be the numerical error separating the exact and the calculated norm value. Then,

$$\|TW - W\| \leq \hat{\|}\widehat{T}W - W\hat{\|} + \|\epsilon^M\| + \|\epsilon^I\| + \epsilon^N \equiv E^c. \tag{7}$$

The bounds on the distance between any $W \in C(X \times Z)$ and the true value function is now obtained in a numerical setting by a combination of (5) and (7):

PROPOSITION 1: For all $(x,z) \in X \times Z$

$$\frac{W(x,z) - E^c}{1-\beta} \equiv \underline{V}(x,z) \leq V(x,z) \leq \overline{V}(x,z) \equiv \frac{W(x,z) + E^c}{1-\beta}.$$

Without loss of generality, it is assumed below that E^c is non-zero and finite.

Although little can be said about the size of the three error terms in (7) in general, they are dealt with easily in many standard cases. If, for instance, the optimization problem in (6) is concave as assumed below, the optimization error is usually negligible. The integration error should be nothing more than rounding errors, if z' takes a finite number of values. Even for distributions with continuous support, the integration error is small and often safely ignored by the literature when appropriate numerical methods are used. Likewise, the norm error is often negligible.[3]

[3]The effect of integration errors is neglected by the *bounded rationality measure* suggested by Judd (1992). See also Santos (2000, p. 1396). The size of the numerical norm errors does not receive explicit attention by Judd (1992) or Santos (2000, p. 1394).

4. NUMERICAL ERROR BOUNDS FOR THE POLICY FUNCTION

Often, the precision of the calculated policy $\hat{G}$ implied by (6) attracts more interest than the precision of the value function approximation. By imposing concavity (not necessarily strict) on the return function and the value function approximation W as well as convexity of the admissible policy space, such bounds can be calculated based on the value function bounds of the previous section.

ASSUMPTION 5: For each $z \in Z$, $F(\cdot,\cdot,z) : A_z \to \mathbb{R}$ satisfies

$$F\big(\theta(x,y)+(1-\theta)(x',y'),z\big) \geq \theta F(x,y,z)+(1-\theta)F(x',y',z),$$
$$\text{all } \theta \in (0,1), \quad \text{and all } (x,y),(x',y') \in A_z.$$

ASSUMPTION 6: For all $z \in Z$ and all $x, x', \in X$

$$y \in \Gamma(x,z) \quad \text{and} \quad y' \in \Gamma(x',z) \text{ implies}$$
$$\theta y+(1-\theta)y' \in \Gamma\big(\theta x+(1-\theta)x',z\big), \quad \text{all } \theta \in [0,1].$$

ASSUMPTION 7: For each $z \in Z, W(\cdot,z) : X^l \to \mathbb{R}$ is bounded and concave.

Since the approximation is not required to be *strictly* concave, the popular piecewise linear approximation is allowed. Note also, that the contraction bounds of Proposition 1 inherit the concavity of W since E^c is a constant.

Below, the idea is to bound the derivatives of the problem and then use these bounds in connection with the first order conditions for the optimum to bound the area in which the optimal (but unknown) policy belong. Since differentiability with respect to endogenous state variables is not required, derivative bounds should be interpreted as bounds on the subdifferentials instead.[4]

4.1. DERIVATIVE BOUNDS

Consider, for each $(x,z) \in X \times Z$ and an arbitrary vector $\Delta x \in \mathbb{R}^l$, the function $V(x+\lambda\Delta x,z)$, $\lambda \in \mathbb{R}$ defined at the $x_0+\lambda\Delta x$ -selection of the state space, $X_\Delta \subset X$ and let $\partial V_\Delta(x,z)$ denote the directional subdifferential with respect to Δx. Finally, let $\overline{x} = \sup(X_\Delta)$ and $\underline{x} = \inf(X_\Delta)$ denote the upper and lower bounds of the relevant state space selection.

PROPOSITION 2: Let Assumptions 1–7 hold. Consider an arbitrary point $(x,z) \in X \times Z$ and let $\underline{V}$ and $\overline{V}$ be defined by Proposition 1. Let $\underline{H}$ and $\overline{H}$, both $X_\Delta \to \mathbb{R}$, be the two supporting hyperplanes to $\underline{V}(x+\lambda\Delta x,z)$ for $\lambda \geq 0$ and $\lambda \leq 0$ respectively, both passing through point $((x,z),\overline{V}(x,z))$; see Figure 1. Let $\underline{\partial V_\Delta}$ and $\overline{\partial V_\Delta}$ denote the first order derivatives of $\underline{H}$ and $\overline{H}$ respectively. Then[5]

$$\underline{\partial V_\Delta} \leq \partial V_\Delta(x,z) \leq \overline{\partial V_\Delta}.$$

[4] As noted by Rockafellar (1970, Sec. 30), the term *superdifferential* might be more appropriate for concave functions.

[5] Since $\underline{H}$ and $\overline{H}$ are vertical in case $x = \overline{x}$ and $x = \underline{x}$ respectively, define in these cases the values of the bonds in accordance with their limiting values $\underline{\partial V_\Delta}(\overline{x},z) = -\infty$ and $\overline{\partial V_\Delta}(\underline{x},z) = \infty$.

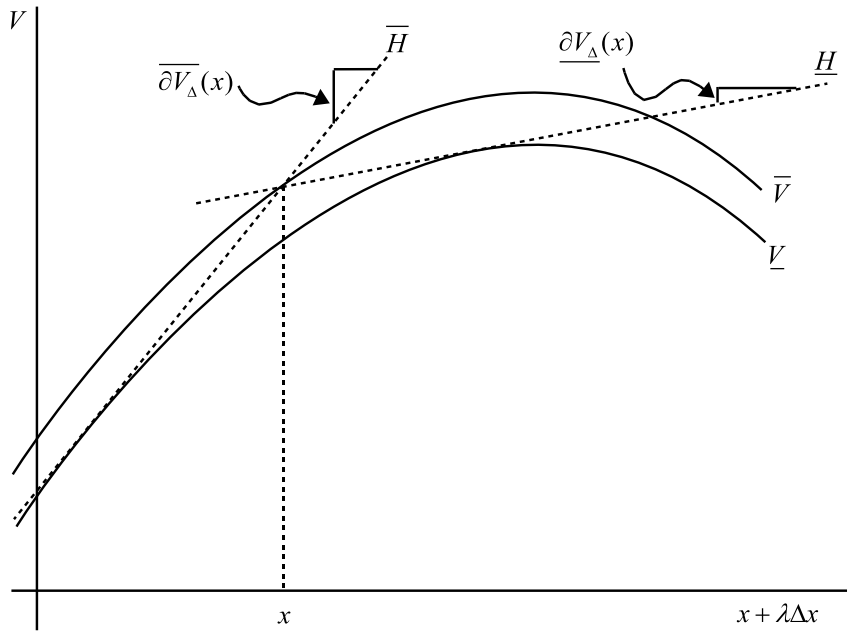

FIGURE 1. Proposition 2: ∂V is bounded

PROOF: See appendix A.1

Since $\underline{V}$ and $\overline{V}$ are concave and available without numerical errors, the appropriate hyperplanes in Proposition 2 can be determined without significant numerical errors.

Note that the derivative bounds, unlike the value function bounds, are local and their quality will differ over the state space. Although local, the bounds of Proposition 2 depend on the quality of the global bound through Proposition 1. If V approximates W badly in even a small area of the state space, the quality of the derivative bounds will then be affected for the entire state space.

The following lemma states that even when $\partial V_\Delta(x)$ is multi-valued, non-continuous, and only weakly decreasing, the bounds provided by Proposition 2 are single-valued, continuous and strictly decreasing. Such properties are obviously useful in numerical implementations.

LEMMA 1: The derivative bounds defined in Proposition 2 are single-valued and strictly decreasing at X_Δ and continuous at the interior of X_Δ.

PROOF: See appendix A.2

4.2. POLICY BOUNDS

To investigate the first order conditions for the optimal policy, define the function $v(y)$ as:

$$v(y) = F(x, y, z) + \beta \int V(y, z')Q(z, dz') \tag{8}$$

where the dependence of (x, z) is implicit. In optimum $v(g) = TV = V$, and since $v(g)$ is concave, the usual first order conditions characterize the optimal policy. For non-differentiable problems, first order conditions must be stated in terms of subdifferentials with respect to g:

$$0 \in \partial v_{\Delta}(g), \quad \text{all } \Delta g \in \mathbb{R}^l.$$

The true values of ∂v_{Δ} are, of course, unknown since V on the right hand side of (8) is unknown. However, the bounds of Proposition 2 together with the derivatives of F provide bounds on $\partial v_{\Delta}(g)$ and, hence, the optimal policy.

For most economic applications, F will be differentiable with respect to the policy variable, but for the sake of generality, differentiability of F is not assumed below. Therefore, let ∂F_{Δ} denote the subdifferential of F with respect to Δg and let $\overline{\partial F_{\Delta}}$ and $\underline{\partial F_{\Delta}}$ denote the upper and lower values. With this notation in place, the bounds on $\partial v_{\Delta}(g)$ are defined for all $\Delta g \in \mathbb{R}^l$ as:[6]

$$\begin{aligned} \overline{\partial v_{\Delta}}(g) &= \overline{\partial F_{\Delta}}(x, g, z) + \beta \int \overline{\partial V_{\Delta}}(g, z')Q(z, dz') \\ \underline{\partial v_{\Delta}}(g) &= \underline{\partial F_{\Delta}}(x, g, z) + \beta \int \underline{\partial V_{\Delta}}(g, z')Q(z, dz'). \end{aligned} \tag{9}$$

The implication for the optimal policy is stated below in a proposition without proof.

PROPOSITION 3: Let Assumptions 1–7 hold and let $\overline{\partial v_{\Delta}}(g)$ and $\underline{\partial v_{\Delta}}(g)$ be given by (9). Then $g(x, z) \in G(x, z)$, $(x, z) \in X \times Z$, if and only if for all $\Delta g \in \mathbb{R}^l$

$$\underline{\partial v_{\Delta}}(g) \leq 0 \leq \overline{\partial v_{\Delta}}(g),$$

except for possible boundary solutions with respect to Γ.

So far, nothing has been assumed about differentiability. However, if F is continuous differentiable with respect to y, the policy bounds $\underline{g}$ and $\overline{g}$ are simply determined by standard first order conditions like

$$\begin{aligned} F_{y^i}(x, \underline{g}, z) + \beta \int \underline{\partial V_{y^i}}(\underline{g}, z')Q(z, dz') = 0, \\ F_{y^i}(x, \overline{g}, z) + \beta \int \overline{\partial V_{y^i}}(\overline{g}, z')Q(z, dz') = 0, \end{aligned} \tag{10}$$

for $i \in \{1, 2, \ldots, l\}$, except for possible corner solutions w.r.t. Γ. Note that the policy bounds (like the derivative bounds) are local and can be calculated for a single point of interest once E^c is determined.

[6]Note that $\partial \int V_{\Delta} \neq \int \partial V_{\Delta}$, since ∂V_{Δ} might be multi-valued when V is not differentiable. On the contrary, the bounds provided by Proposition 2 are single-valued and $\int \overline{\partial V_{\Delta}}$ and $\int \underline{\partial V_{\Delta}}$ are well-defined. Moreover, they are upper and lower bounds for $\partial \int V_{\Delta}$.

5. TWO EXAMPLES

First, this section shows how the bounds suggested above can be used on linear models. Then, the bounds are compared with the bounds suggested by Santos (2000) and Maldonado and Svaiter (2001) using a standard growth model. The two models considered are both special cases of the following setup.

Consider an infinitely lived agent who wants to maximize expected discounted utility from consumption c,

$$\max E_0 \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\gamma} - 1}{1-\gamma}, \quad 0 \leq \gamma, \quad 0 < \beta < 1. \tag{11}$$

The agent receives income from a production which is assumed to be linear in the level of capital, Ak_t, where $A > 0$ is a constant and capital k_t is assumed to depreciate fully each period. The production is sold at one of two prices: Either the production is sold on the world market, in which case the price is one, $p_t = 1$, and the model is a version of the AK-model,

$$c_t = Ak_t - k_{t+1}.$$

Or the production is sold on the home market with a price function equal to $p_t = k_t^{\alpha-1}$. In this case the model is similar to a simple one-sector growth model,

$$c_t = Ak_t^{\alpha} - k_{t+1}, \quad 0 < \alpha < 1. \tag{12}$$

The price process is assumed to be exogenous and Markov, and will be described explicitly below. Following Santos and Vigo-Aguiar (1998), the initial parameter values are fixed at $\beta = 0.95$, $A = 5$, $\alpha = 0.34$.

Due to the simple price structure, the state of the model is described by the level of capital and an indicator of which market is currently the relevant one. Based on the state of the model, the agent chooses k_{t+1} optimal according to (11).

THE NUMERICAL APPROXIMATION METHOD

The special cases considered below both provide closed form solutions for value functions as well as policy functions. The numerical approximations used below to test the bounding procedures are splines of various order fitted to the true value function under the L_2-norm based on a finite number of approximation points (least square). The spline pieces are defined over equal-sized subintervals. The value function representation has been chosen, because the implied policy function is easily calculated with very high precision. The calculation of the implicit value function from a policy function approximation is usually much more complicated, see also Section 5.5. By using the true value function to fit the numerical approximation, the issue of solution algorithm is deliberately avoided while the issue of the size of the numerical error term remains as long as the true value function is not an element of the approximation basis used. Finally note, that due to the simple price structure, the value function can be approximated with two one-dimensional approximations; one for each state of market.

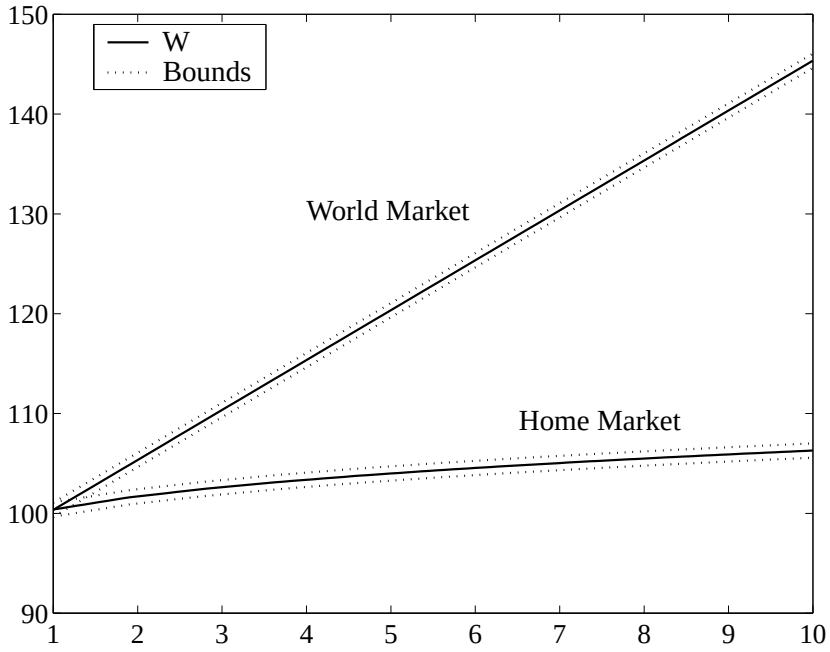

FIGURE 2: Value function approximations, world market and home market.

5.1. A MODEL WITHOUT STRICT CONCAVITY

Assume linear utility $\gamma = 0$ and assume that p_t follows a white noise process with the following distribution,

$$p_t = \begin{cases} 1 & \text{with probability} \quad q \quad = 0.1 \\ k^{\alpha-1} & \text{with probability} \ (1-q) = 0.9. \end{cases}$$

For each market, the capital state space is restricted to $[1, 10]$. Below, piecewise linear splines will be used to approximate the value function. Hence, the model is clearly not strictly concave. When the produced goods can be sold on the world market, utility, production and return on capital are linear whereas the value function approximation is piecewise linear. Figure 2 shows a piecewise linear value function approximation with 10 subintervals and the resulting contraction error bounds of Proposition 1 for each market state.[7] In the world market state, the value of capital is clearly linear. Still, the possibility that future production might be sold on the home market with a decreasing return on capital, causes an interior optimal policy. Note that with linear utility, full depreciation of capital and a white noise price process, the optimal policy is independent of the state of the model. As a consequence of this state independence, the bounds on the first order derivative and the optimal policy shown in Figure 3 are valid for all states.

Figure 3 is based on the same settings as above except that the value function approximation is constructed using 100 subintervals for each market state. Despite higher pre-

[7]Optimization and integration errors are safely ignored. The sup-norm calculation is based on a number of equally spaced points. The error arising from this procedure is ignored.

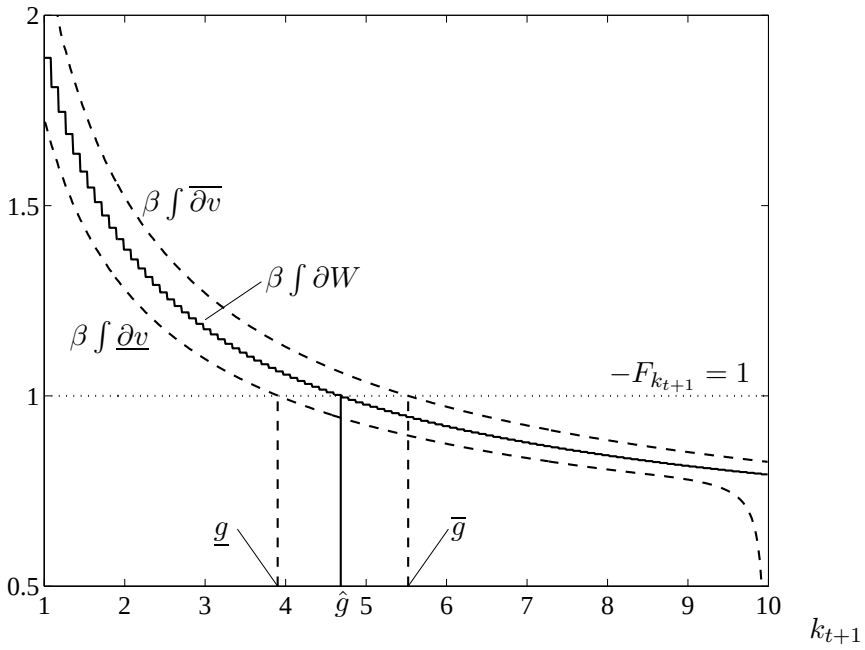

FIGURE 3: Derivative and policy bounds

cision, it is clear from the figure that the first order derivative of the approximation W is not smooth. As predicted by Lemma 1, the derivative bounds are nevertheless smooth at the interior of X. Since the marginal utility of investments is continuous (equal to -1), the policy bounds $\underline{g}$ and $\overline{g}$ are determined by (10).

Note that interior solutions are not required and that restrictions imposed by Γ are easily dealt with. If these restrictions are tighter than the bounds suggested by Figure 3, $\underline{g}$ and $\overline{g}$ are simply adjusted accordingly. In the present case, $\Gamma = [1, 10]$ does not affect the bounds, however.

5.2. A STANDARD TEST MODEL

The model in Section 5.1 does not satisfy the assumptions of the error bounds suggested by Santos (2000) and Maldonado and Svaiter (2001). In order to compare the various bounding procedures, consider instead a model where the produced goods are always sold on the home market according to (12) and where the utility of consumption is logarithmic, $\gamma = 1$. Finally, by assuming a state space of capital equal to $[0.1, 10]$, the model is identical to the growth model used by Santos and Vigo-Aguiar (1998) to test the properties of bounding procedures. For the sake of formality, Γ is chosen such that consumption is bounded below by a small positive number in order to avoid infinite utility. Closed form expressions for both the value function, $v(k) = d_0 + d_1 \ln(k)$,[8] and the optimal policy function, $g(k) = \alpha\beta k^{\alpha}$, are available. Note from the closed form of the value function that an accurate numerical approximation at the interval $[0.1, 1)$ is difficult to achieve, since the size of the higher order derivatives explodes due to the asymptotic behavior of

[8]Where $d_0 = (1-\beta)^{-1}\big(\ln(1-\alpha\beta) + \alpha\beta(1-\alpha\beta)^{-1}\ln(\alpha\beta)\big)$ and $d_1 = \alpha(1-\alpha\beta)^{-1}$

the logarithm near zero. As a consequence, a 10th order 4 subinterval spline was used for Table 1 in order to obtain acceptable precision.

Table 1 shows the effectiveness of the various bounding procedures measured as the difference between the true policy function and the policy function implied by the value function approximation.

The first line reports the figures for the benchmark settings. When measured in the sup-norm, the table shows that the bounds implied by Proposition 3 above are 31.1 times looser than necessary. The effectiveness of the bound suggested by Maldonado and Svaiter is lower with 136.2. With a figure of 2275, the bound suggested by Santos (2000, Theorem 3.3) is less effective in the present case. The asymptotic bound suggested by Santos (2000, Theorem 3.5) is even less effective.

Since the Proposition 3 bound is local for a given value error bound, the effectiveness of the bound need not be measured in the sup-norm. The L_2-norm figures show the average size of the upper and lower bound. They are both comparable to the sup-norm error. The size of the error might even be calculated for a single point of interest as, for instance, the steady state. Table 1 shows that the bounds in steady state are tighter than the average bounds.

To give an indication of the precision of the solution across various parameter choices, the last column of Table 1 shows the size of the true sup-norm error relative to the true sup-norm error of the benchmark case. The absolute sup-norm error value in the benchmark case is 0.1156.

A figure of direct importance to the bounds described by Santos and Maldonado and Svaiter is the concavity of the model, whereas the influence of the concavity on the bound in Proposition 3 is less direct. The importance of concavity can be investigated by varying α. The effect on the Santos bound is as expected: Low α-values (high concavity) causes more effective bounds, whereas the effect on the Maldonado and Svaiter bound and the Proposition 3 bound is less monotonic. The L_2 and the steady state bounds show that the effectiveness of the P3 upper bound tends to increase with concavity whereas the lower bound shows the opposite causality.

It is well known that low discount factors destroy the effectiveness of the contraction error bounds used by Maldonado and Svaiter and Proposition 3. The figures in Table 1 show that this is also the case for the Santos bounds, which is in accordance with the findings by Santos (2000).

Table 1 shows that if the size of the state space is reduced, the relative effectiveness of the bounds suggested by Santos (2000) increases significantly. This effect is connected to the quality of the approximation. As discussed, the most problematic interval to approximate is $[0.1, 1)$. Once this interval is disregarded, the true policy function error drops to only 0.27% of the error of the benchmark case (see last column). If, on the other hand, the interval between 5 and 10 is disregarded, the error only drops to 24.72%. The importance of the quality of the approximation on the relative effectiveness of the various bounding procedures is looked at more closely below.

5.3. The Asymptotic Properties of the Bounding Procedures

If the precision of a single numerical solution is investigated, the tightness of the error bound procedure is essential. If, however, different numerical approximations are to be

TABLE 1

EFFICIENCY OF POLICY FUNCTION BOUNDING PROCEDURES

	sup-norm				P3: L_2-norm		P3: Steady State		True
Setup	Santos	Asymp	M&S	P3	Low	High	Low	High	sup-error
Benchmark	2275	1.1E5	136.2	31.10	24.03	20.67	19.08	15.75	100% =0.1156
α = 0.17	756.7	39590	146.9	48.86	41.85	18.29	37.05	16.51	101.5%
α = 0.66	17166	9.8E5	243.7	49.48	13.77	29.30	8.119	16.97	99.53%
α = 0.83	1.0E5	7.0E6	398.3	72.60	9.060	39.63	4.493	19.18	88.33%
β = 0.9	1128.7	27297	93.38	58.18	18.75	16.96	15.03	12.98	102.5%
$\beta = 0.99$	11481	2.7E6	312.5	47.32	36.18	22.68	28.48	17.15	97.99%
$k \in [1, 10]$	928.8	43653	1652	490.8	388.2	387.6	287.2	286.8	0.2678%
$k \in [0.1, 5]$	1435	1.1E5	172.9	70.13	53.76	50.16	52.33	49.34	24.72%

Notes: All figures, except the last column, represent the ratio of the size of the maximum error according to the bound over the size of the true error. The last column shows the size of the true sup-norm error relative to the true error in the benchmark case. "Santos" refers to Santos (2000, Theorem 3.3), (with H = 1 assumed). "Asymp" refers to Santos (2000, Theorem 3.5). "M&S" refers to Maldonado and Svaiter (2001, "Main Theorem"). "P3" refers to the bounds suggested by Proposition 3 above. The "L_2-norm" figures refer to Proposition 3 bounds according to the L_2-norm. Results are available for lower and upper bounds. The "Steady State" figures represent Proposition 3 bounds in steady state. All figures are based on a 10th order 4 subintervals spline approximation (i.e. 13 free parameters) of the true value function (fitted in the L_2-norm) and the implicit policy function. The benchmark settings are: $A = 5$, $\alpha = 0.34$, $\beta = 0.95$, $\gamma = 1$, $q = 0$, and $k \in [0.1, 10]$. In the other cases, A is adjusted to maintain the original steady state value of capital.

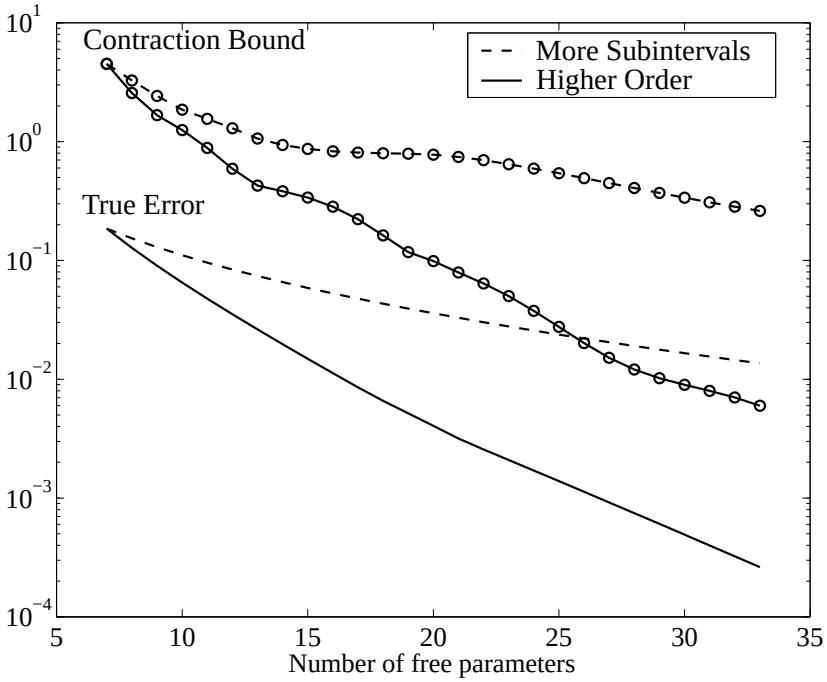

FIGURE 4: Value function errors and bounds

compared, monotonicity of the bounds might be more useful: If one approximation is better than another, this fact should be reflected by the error bounds.

Below, the value function of the benchmark case from above is approximated with a four subintervals cubic spline, raising the following question. Which procedure would be the most effective for reducing the approximation error: 1) increasing the number of subintervals while keeping the cubic order fixed, or 2) increasing the spline order while keeping the number of subinterval fixed? Such questions might be answered by the bounding procedures analyzed above even when the true solution is unknown.

Figure 4 compares the actual sup-norm error of the value function approximations to the size of the bound implied by the contraction error bound in (3), both shown as functions of the number of free parameters used in the approximation basis. The starting point (a four subinterval cubic spline) has seven free parameters. Each time a subinterval is added or the order of the approximation is increased, one free parameter is added to the approximation. The figure shows that the extra parameters are better spent on higher orders than more subintervals.[9] This information is clearly reflected by the contraction error bound.

CONVERGENCE RATES, HIGHER ORDER

For sufficiently small state spaces the Taylor series approximation result suggests that approximation errors will converge to zero at an exponential rate when the polynomial order is increased. More formally, $\|\epsilon_n\| = \mathcal{O}(\exp(\theta n))$ where $\|\epsilon_n\|$ is the sup-norm ap-

[9]Note, however, that higher order polynomials are more expensive to evaluate with respect to computer time.

proximation error, n is the order and θ is the exponential rate of convergence defined as

$$\theta_{n,n-j} = \frac{\log(\|\epsilon_n\|) - \log(\|\epsilon_{n-j}\|)}{n - (n-j)}$$

when changing the order of the approximation from $n-j$ to n. According to the Taylor series result, $\theta = -1$ if the approximated function is analytical and the state space is sufficiently small.

The first two lines of Panel A in Table 2 show the values of $\theta_{20,4}$ for the actual error and the contraction error bound of (3) when the settings correspond to those of Figure 4, except for variations in the size of the state space. For state spaces including the $[0.1, 1)$ interval, the exponential convergence rate is clearly lower than if this interval is excluded. In cases where the closed form solution is unknown, such information is, of course, almost impossible to obtain without the use of error bounds. Fortunately, the contraction error bounds reveal very detailed the convergence rates for all sizes of the state space.

CONVERGENCE RATES, MORE SUBINTERVALS

When the number of subintervals is increased for the cubic spline approximation, we might expect a polynomial rate of convergence. More formally, $\|\epsilon_n\| = \mathcal{O}(n^\gamma)$, where n is the number of subintervals and γ is the polynomial rate of convergence defined as

$$\gamma_{n,n-j} = \frac{\log(\|\epsilon_n\|) - \log(\|\epsilon_{n-j}\|)}{\log(n) - \log(n-j)}$$

when changing the number of subintervals from $n-j$ to n. Based on theoretical results, a polynomial rate of convergence of -4 might be expected, see e.g. de Boor (1978).

Line three and four of Panel A in Table 2 report the values of $\gamma_{20,4}$ for the actual error and the contraction error bound of (3) when the number of subintervals are increased. As for the exponential convergence rates it can be concluded that the true polynomial convergence rate is also fairly accurately predicted by the error bounds.

5.4. CONVERGENCE OF POLICY FUNCTION ERRORS

Consider now a convergence analysis of the policy function errors, similar to the one above for value function errors.

Figure 5 shows the policy function equivalent to Figure 4. Note first, that all bounds are monotone in the true error for all but a few observations. Note also that the ranking of the three bounding procedures reflect the results for the benchmark case in Table (1).[10] The convergence rate of the Santos bound appear superior to the Maldonado and Svaiter bound and the Proposition 3 bound, however. This impression is supported by Panel B of Table 2.

First, the exponential convergence rates $\theta_{20,4}$ arising from increased orders are shown. The rates for the true policy error are almost the same as the corresponding value function rates. This connection seems independent of the size of the state space. The Santos bound is the only one to show the convergence rate of the true error fairly accurately. The Maldonado and Svaiter bound and the Proposition 3 bound show convergence rates of

[10] The asymptotic error bounds of Santos (2000, Theorem 3.5) are excluded.

TABLE 2

CONVERGENCE RATES

		$k \in [0.1, 10]$	$k \in [0.1, 5]$	$k \in [1, 10]$	$k \in [1, 5]$
		Panel A: Value Function			
$\theta_{20,4}$	True error	-0.280	-0.376	-0.811	-1.14
	Contraction	-0.281	-0.352	-0.815	-1.11
$\gamma_{20,4}$	True error	-1.18	-1.57	-2.95	-3.49
	Contraction	-1.21	-1.47	-3.18	-3.40
		Panel B: Policy Function			
$\theta_{20,4}$	True error	-0.262	-0.310	-0.792	-1.05
	Santos	-0.240	-0.311	-0.731	-1.03
	M&S	-0.141	-0.176	-0.408	-0.556
	Prop. 3	-0.0976	-0.123	-0.403	-0.556
$\gamma_{20,4}$	True error	-2.02	-2.36	-2.61	-2.87
	Santos	-1.17	-1.66	-1.96	-2.59
	M&S	-0.604	-0.735	-1.59	-1.70
	Prop. 3	-0.302	-0.253	-1.54	-1.70

roughly half the size of the true rate. This result seems connected to the asymptotic results of Santos (2000), who shows that the approximation error of the policy function is of the same order of magnitude as the size of the Euler equation residuals upon which the Santos bound is based. However, the asymptotic relation between policy function errors and the value function errors (upon which the contraction bound is based) is only of a square-root order of magnitude.[11]

The polynomial convergence rates $\gamma_{20,4}$ for the policy function approximation errors in Panel B, Table 2 show much the same raking among the error bounds as in the exponential case. The Maldonado and Svaiter and the Proposition 3 bounds underestimate the convergence rate, whereas the Santos bound is closer to the true convergence rate. In the polynomial case, however, the Santos bound seems less precise than in the exponential case. For the benchmark state space, for instance, the true convergence rate of -2.02 is significantly underestimated by the Santos bound that shows a convergence rate of only -1.17.

Note also from Table 1 that for small state spaces (high precision), the convergence rate

[11] This last argument, however, should be seen in the light of the comments given in Section 5.5.

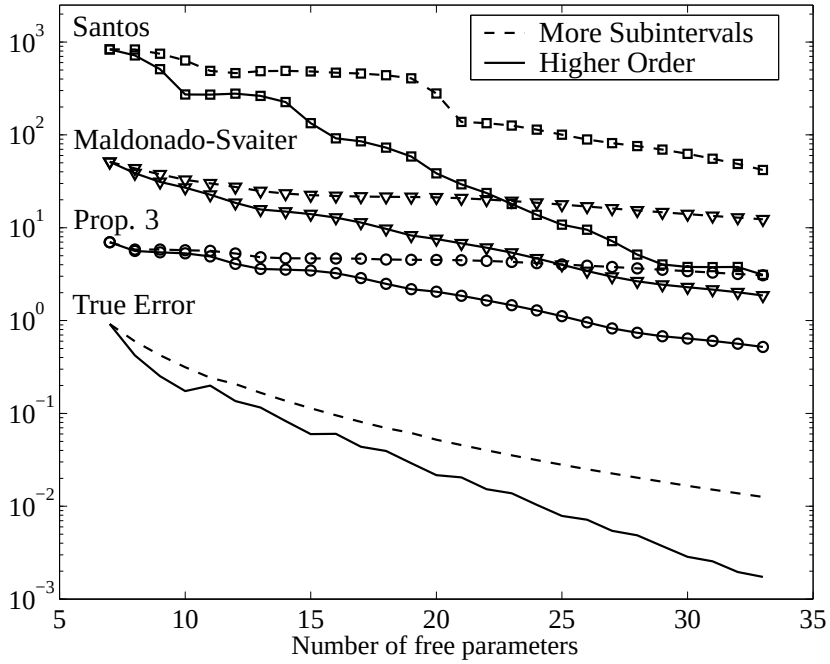

FIGURE 5: Policy function errors and bounds

of the Maldonado and Svaiter bound and the Proposition 3 bound are very close. This suggests that the two bounds have the same asymptotic convergence rates.

Finally, it is worth mentioning that despite the higher convergence rate of the Santos bounds, Figure 5 shows that both the Maldonado and Svaiter bound as well as the Proposition 3 bound even, for quite accurate solutions, are tighter than the Santos bound.

5.5. ALTERNATIVE VALUE FUNCTION BOUNDS

For a given precision of the value function approximation, the policy bounds suggested by Santos (2000) and Maldonado and Svaiter (2001) are similar. The reason why they differ are their different approaches on how to bound the precision of value functions. Where Maldonado and Svaiter (and Proposition 3) use the contraction mapping bound to bound the error of the calculated value function W, Santos bounds the precision of the value function $V_{\hat{g}}$ implied by the calculated policy $\hat{g}$. Given strong regularity conditions, a powerful bound can be derived for the implied value function

$$\|V - V_{\hat{g}}\| \leq C\,\epsilon^2 \tag{13}$$

where ϵ is the sup-norm Euler equation error and C is a constant, see Santos (2000, Theorem 3.3). The quadratic order of convergence in (13) is the reason for the superior asymptotic results of the policy bounds of Santos.

The bound in (13) has an important precision interpretation, since it provides the maximum utility loss incurred by following the approximate policy $\hat{g}$ instead of the true policy g.

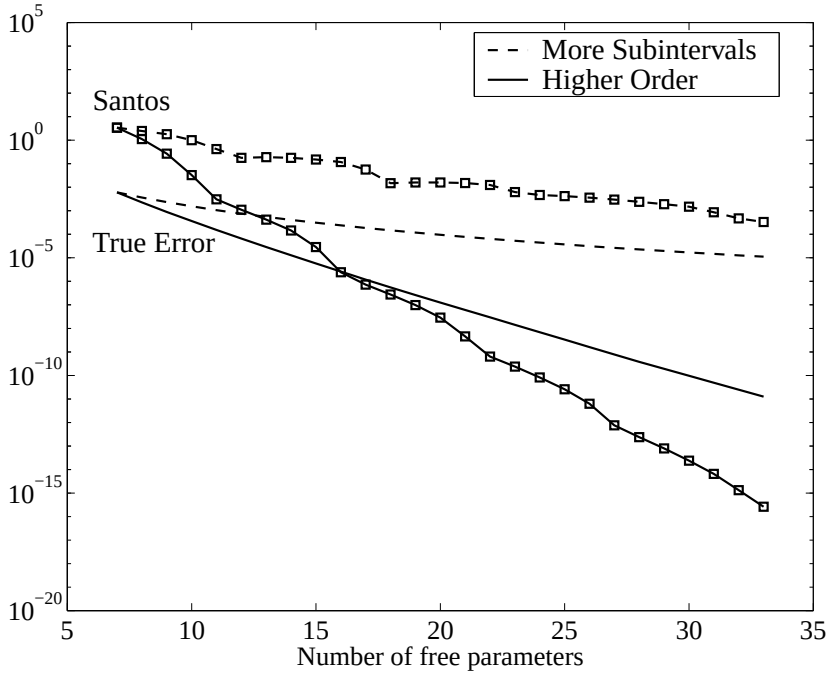

FIGURE 6: Value function errors and bounds

If the precision of a calculated value function is of interest, however, the limitations of the result in (13) should be recognized. A hypothetical example might illustrate the point: Assume that the approximate value function is equal to the true value function plus a constant $W = V + c$ and assume that the optimal policy is characterized by the Euler equation. Since the constant does not affect the Euler equation, the approximate policy implied by W is equal to the true policy function, $\hat{g} = g$.[12] Since there are no violations of the Euler equation, the right hand side of (13) is zero. This result is in accordance with the left hand side since the value function implied by the approximate policy is equal to the true policy function, $V_{\hat{g}} = V$. However, the initial value function approximation clearly differs from the true value function.

Figure 6 shows that this point is of practical importance. The figure shows the true value function error $\|W - V\|$ based on the same settings as used in Figure 4 except that the state space is reduced to $k \in [1, 10]$. In addition, the bound implied by (13) is shown. The lack of coherence between the two measures is clearly illustrated by the fact that the size of the bound for $\|V - V_{\hat{g}}\|$ converges at a higher rate than the size of the true error $\|V - W\|$, eventually falling below the true error.

As another point of practical importance note that although $\hat{g}$ is available without significant errors, it is usually quite difficult to calculate $V_{\hat{g}}$, especially in stochastic settings.

These arguments suggest that the bound in (13) contain more information about the precision of the policy function $\hat{g}$, than about the initial value function approximation W, which might have lead to the approximate policy in the first place.

[12] Except for optimization errors which can often be neglected in practice.

6. CONCLUSION

A numerical solution to a model within the class of models described above can be represented as a value or a policy function.

If the precision of a calculated value function is of interest, it was argued in Section 3 that the well-known contraction mapping error bound, possibly corrected with the appropriate numerical error terms, is a valid approach for a very wide range of problems. The examples in Section 5 showed that the bound is reliable in detecting relative precision and in predicting convergence rates. Moreover, it was argued in Section 5.5 that alternative error bounds might not provide the wanted figure.

If the precision of a given policy function approximation is of interest, the choice of precision measure is less clear. If the functional forms of the model or the approximation functions display linear elements, the bounds suggested by Proposition 3 above are currently the only option. If the model is well-behaved (in the sense of Santos (2000)) the choice of measure seems to depend on the purpose. If tightness is needed, more general bounds as those of Maldonado and Svaiter (2001) and Proposition 3 might outperform the bounds suggested by Santos (2000) for numerical solutions of low to medium precision. For very high precision solutions, however, the bounds of Santos are likely to dominate. This is also the case if estimates of convergence rates are needed. With regard to the examples investigated, all three bounds do a fair job in ranking different approximations according to their relative precision, but the bounds suggested by Santos are superior in predicting rates of convergence. These properties, however, come at the expense of quite restrictive assumptions on the non-primitives of the model, which can be impossible to verify in practice.

Finally, it should be noted that the choice of bounding method might be dictated by the choice of solution method. Value function based approximation methods leave open all of the alternatives described. The popular Euler equation based approximation methods provide only a policy function approximation that cannot easily be converted into the implied value function approximation. In this situation the bounds suggested by Santos are currently the only option if upper bounds on numerical errors are wanted.

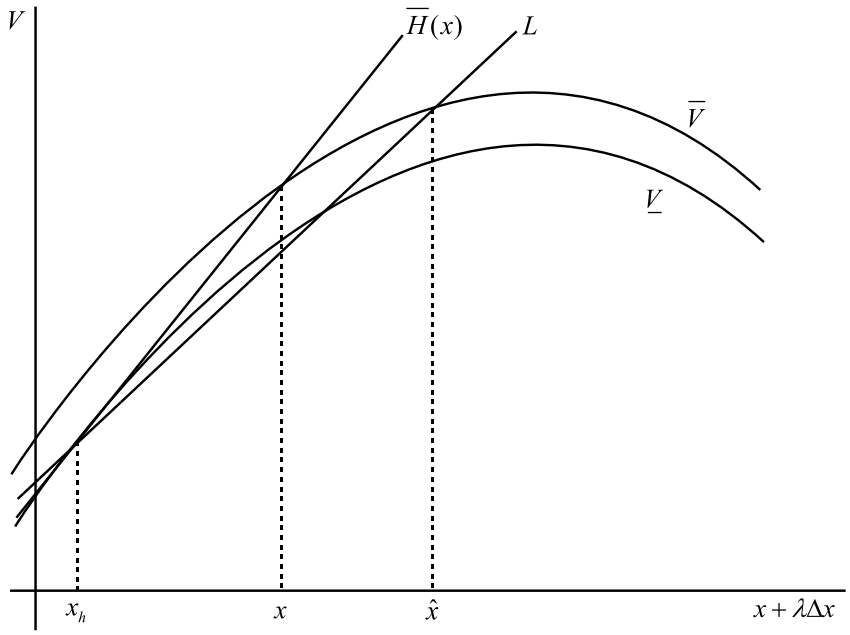

FIGURE 7: Decreasing $\overline{\partial V_\Delta}$

APPENDIX A PROOFS

A.1 PROPOSITION 2

PROOF: Consider the $\overline{\partial V_\Delta}$ part of the proposition and let x be different from $\underline{x}$. Since the true value of V in (x, z) is unknown, consider first an arbitrary value a within the known bounds $a \in [\underline{V}(x, z), \overline{V}(x, z)]$ and let H_a be the supporting hyperplane to $\underline{V}(x + \lambda\Delta x, z)$, $\lambda \leq 0$ that passes through the point (x, a). Such a unique hyperplane exists since $\underline{V}$ is concave and since $x \neq \underline{x}$ is assumed. The slope of H_a is an upper bound on ∂V_Δ if a is the true value of the value function V in x. If not, the concavity of V will make V stay below H_a at X_Δ and eventually violate the lower bound of Proposition 1, $\underline{V}$, as x approaches $\underline{x}$

Naturally, the value of $V(x, z)$ is unknown, so the least favorable a must be found to bound $\overline{\partial V_\Delta}$. Since the derivative of H_a increases in a, the hyperplane associated with $a = \overline{V}(x, z)$ (denoted $\overline{H}$) provides an upper bound on $\partial V(x, z)$, see Figure 1.

Now consider the case where $x = \underline{x}$. No non-vertical hyperplane fulfilling the description above exists. It is easily shown, however, that the limiting value of the slope is plus infinity.

The $\underline{\partial V_\Delta}$ part is proved accordingly. $Q.E.D.$

A.2 LEMMA 1

PROOF: Consider first the $\overline{\partial V_\Delta}$ part of the Lemma, and let the dependence of z be suppressed notationally.

Single-valued: Straightforward since the hyperplane is unique.

Strictly decreasing: Choose two points $x, \hat{x} \in X_\Delta$, where $\hat{x} = x + \lambda\Delta x$ for $\lambda > 0$, see Figure 7. To prove that the upper bound of the value function derivative in $\hat{x}$ is strictly lower than in x, consider the hyperplane $\overline{H}(x)$ from Proposition 2 associated with x. Let x_H be an arbitrary contact

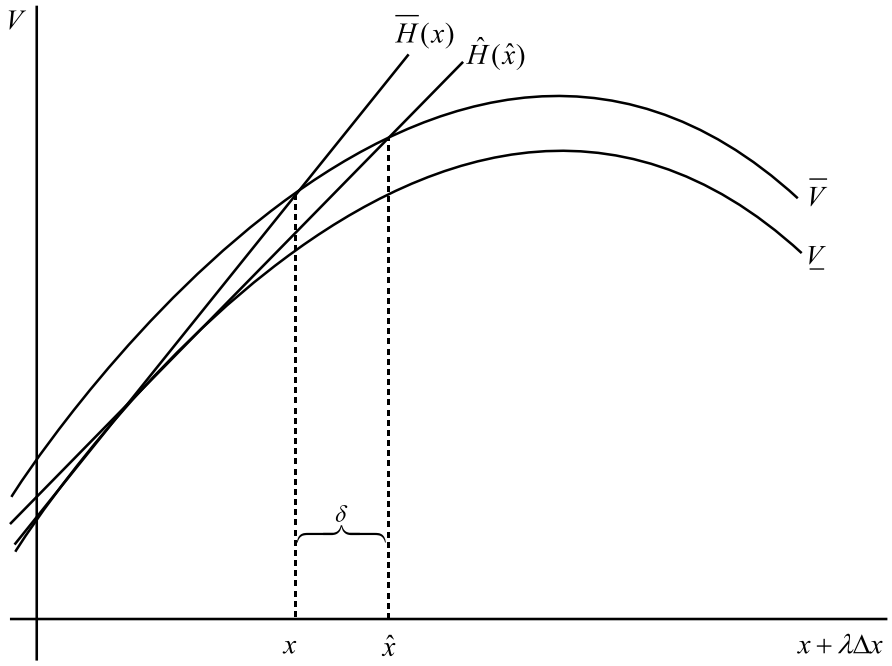

FIGURE 8: Continuous $\overline{\partial V_\Delta}$

point between $\underline{V}$ and $\overline{H}(x)$, (there might be more than one if $\underline{V}$ is linear). Consider now the line L intersecting $\overline{H}(x)$ in $(x_H, \underline{V}(x_H))$ as well as the point $(\hat{x}, \overline{V}(\hat{x}))$. Since $\overline{\partial V_\Delta}(x) > \partial\overline{V}(x)$ when $E^c > 0$, the point $(\hat{x}, \overline{V}(\hat{x})$ is located strictly below $(\hat{x}, \overline{H}(\hat{x}))$. Therefore, the derivative of L is strictly lower than the derivative of $\overline{H}(x)$, $\partial L < \partial\overline{H}(x)$. However, $\partial\overline{H}(\hat{x}) \leq \partial L$, where the derivative of $\overline{H}(\hat{x})$ (the hyperplan provided by Proposition 2 and associated with $\hat{x}$) is equal to $\overline{\partial V_\Delta}(\hat{x})$. Hence, $\overline{\partial V_\Delta}(\hat{x}) < \overline{\partial V_\Delta}(x)$ and the bound are strictly decreasing.

Continuous at $\operatorname{int} X_\Delta$: First, it is shown that $\overline{\partial V_\Delta}(x)$ is finite for $x \in \operatorname{int} X_\Delta$. Consider an arbitrary $x \in X_\Delta \backslash \underline{x}$ and define $x^{1/2} = (\underline{x} + x)/2 \in X_\Delta$. Now, the derivative of the line L passing trough the points $(x^{1/2}, \underline{V}(x^{1/2})$ and $(x, \overline{V}(x)$ is finite since $\underline{V}$ and $\overline{V}$ are bounded. Then, since the derivative of L bounds the derivative of $\overline{H}(x)$ from above, $\overline{\partial V_\Delta}(x)$ is also bounded above for $x \neq \underline{x}$. Similar arguments can bound $\underline{\partial V_\Delta}(x)$ from below for $x \in X_\Delta \backslash \overline{x}$. Hence, since the lower bound bounds the upper bound from below according to Proposition 2, $\overline{\partial V_\Delta}(x)$ is finite for $x \in \operatorname{int} X_\Delta$.

To show continuity, it must be shown that for every $\epsilon > 0$, there exist a $\delta > 0$ such that $|\overline{\partial V_\Delta}(\hat{x}) - \overline{\partial V_\Delta}(x)| < \epsilon$ for all $|\hat{x} - x| < \delta$. Therefore, consider the supporting hyperplane $\hat{H}$ to $\underline{V}(x), \lambda < 0$ with a slope equal to $\overline{\partial V_\Delta}(x) - \epsilon/2$. Assume that $\hat{H}$ intersects $\overline{V}$ at $\hat{x}$. Then, by the definition of the upper bound given above, $\overline{\partial V_\Delta}(\hat{x}) = \overline{\partial V_\Delta}(x) - \epsilon/2$. Since $\overline{\partial V_\Delta}$ is decreasing, $|\overline{\partial V_\Delta}(\hat{x}) - \overline{\partial V_\Delta}(x)| < \epsilon$ for all $\tilde{x} \in [x, \hat{x}]$. Hence, $\hat{x} - x$ is a valid δ-distance to the right. Along the same lines, a δ-distance can be established to the left.

The $\underline{\partial V}$-part of the proposition can be proved by similar arguments. $Q.E.D.$

REFERENCES

CHRISTIANO, L. J. AND J. D. FISHER (2000): "Algorithms for Solving Dynamic Models with Occasionally Binding Constraints," *Journal of Economic Dynamics and Control*, 24(9):1179–1232.

DE BOOR, C. (1978): *A Practical Guide to Splines*, Springer-Verlag.

DEN HAAN, W. D. AND A. MARCET (1994): "Accuracy in Simulations," *Review of Economic Studies*, 61(1):3–17.

JUDD, K. L. (1992): "Projection Methods for Solving Aggregate Growth Models," *Journal of Economic Theory*, 58:410–452.

MALDONADO, W. L. AND B. F. SVAITER (2001): "On the Accuracy of the Estimated Policy Function using the Bellman Contraction Method," *Economics Bulletin*, 3(15):1429–1445.

RAMSEY, F. P. (1928): "A Mathematical Theory of Savings," *Economic Journal*, 38(152):543–559.

ROCKAFELLAR, R. T. (1970): *Convex Analysis*, Princeton University Press.

SANTOS, M. S. (2000): "Accuracy of Numerical Solutions Using the Euler Equation Residuals," *Econometrica*, 68(6):1377–1402.

SANTOS, M. S. AND J. VIGO-AGUIAR (1998): "Analysis of a Numerical Dynamic Programming Algorithm Applied to Economic Models," *Econometrica*, 66(2):409–426.

STOKEY, N. L. AND R. E. LUCAS, JR. (1989): *Recursive Methods in Economic Dynamics*, Harward University Press, Cambridge, MA. (with the collaboration of Edward C. Prescott).

TAYLOR, J. B. AND H. UHLIG (1990): "Solving Nonlinear Stochastic Growth Models: A comparison of Alternative Solution Methods," *Journal of Business and Economic Statistics*, 8(1):1–17.

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