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On a class of adjustable rate mortgage loans subject to a strict balance principle

by

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Abstract

We describe the background and the basic funding mechanisms for the type of adjustable rate mortgage loans that were introduced in the Danish market in 1996. Each loan is funded separately by tap issuing pass-through mortgage bonds ("strict balance principle"). The novelty is a funding mechanism that uses a roll-over strategy, where long term loans are funded by sequentially issuing short term pass-through bonds, and the first issuer of these loans obtained a patent on the funding principles in 1999. Publicly available descriptions of the principles leave an impression of very complicated numerical algorithms. The algorithms described here show that the essentials can be reduced to a "back of an envelope" complexity.

Keywords: Adjustable rate mortgages, balance principle, patent, yield curve riding

1 Background and motivation

The Danish market for mortgage loans adopted the funding idea of mortgage backed securities more than two hundred years ago. The first credit institution of this kind developed in 1797 after a big fire in Copenhagen in 1795 and the financing needs following this event. In 1850 the first specific law concerning mortgage credit institutions was inaugurated, writing into law and government regulation the structure under which these institutions have functioned since then. Although institutional changes have naturally taken place the core of this part of the financial system has remained quite stable for the entire period.

Although considerable care must be exercised in international comparisons of markets w.r.t. such measures as size and liquidity, the Danish market for bonds and, in particular, mortgage bonds is the largest in the world relative to GDP as well as population size. Roughly 2/3 of the outstanding volume of bonds are issued by mortgage credit institutions and the remaining part is almost exclusively government bonds. Corporate bonds account for only a tiny part of the market. For more recent surveys on the Danish market structure, including market statistics, see e.g. Christiansen et al. (2003), Frankel et al. (2004), Moody (2002) and Realkreditrådet (1998).[1]

Mortgage loans are funded by pooled issues of bonds. The most important characteristics, some of them distinguishing the Danish market from similar markets in other countries, are:

1. The *strict balance principle*. Mortgage credit institutions are only allowed to carry funding risk to a negligible extent. The payments from the debtors must match the payments to the bondholders. Credit institutions charge debtors transaction costs besides a directly specified "contribution", equivalent to an interest rate margin. Both are at a low and very competitive level in comparison with other sources of debt financing. For the average homeowner the borrowing rate differs from the funding rate by a spread of approximately 50bp.

   The credit institutions carry the default risk of debtors, and the bondholders are only in very extreme situations[2] subject to any credit risk. The contribution can partly be interpreted as debtor's payment of an insurance premium for the credit institution's obligation to cover losses from debtor defaults from their equity reserves.

2. Traditionally, this balance principle has been implemented by issuing bonds as *mirror images of the loans*. I.e. a 20 or 30 year annuity loan has been funded by issuing an equivalent amount of 20 or 30 year annuity bonds.

[1] The latter two available from the English language homepage of The Association of Danish Mortgage Banks, http://www.realkreditraadet.dk.

[2] For all practical considerations Danish mortgage bonds are risk free and top-rated investment objects.

3. Long term mortgage bonds are – with few exceptions – *callable at par*. This embedded option is an American style call option, exercisable at each of four annual payment dates, but advance notice is required in order to exercise the option. The only major exception to this has been the market for CPI-indexed mortgage bonds, which is not treated here and which has been used mainly by subsidized building societies and hardly by individual homeowners at all.

4. The *delivery option* on behalf of debtors. Due to the balance principle debtors can only prepay their loan by delivering an equivalent amount of bonds of that particular type which was originally issued in order to finance their loan. The bonds to deliver must be bought in the open market or through exercise of the call option at par. Due to transaction costs and certain tax asymmetries callable bonds can have market prices above par, typically in the range of $2-5\%$.

5. Before each payment date bondholders are notified of the entire amount of repayment in the pool – consisting of both regular repayments and prepayments. For the majority of the bonds with quarterly payments this notification is roughly made in the middle of each quarter. Bonds traded after the notification date only receive interest payment at the first payment date. The owner of the bond at the notification date receives repayments on a pro rata basis[3].

6. Bonds are *tap issued* in pools of bonds with identical characteristics (coupon rate, maturity, payment profile etc.). The pools are usually kept open for tap issue in an opening period of three years. After this opening period, the bonds in a particular pool are perfect substitutes.

7. Mortgage loans are – as the point of departure – *assumable* in case of turnover of the house. Mortgage loans are in a literal sense “real estate finance” related more to the house than to the individual.

The regulation of this market has been extensively used as a macroeconomic policy tool in order to influence aggregate demand through credit market policy in the period starting in the mid 1960’es. It has been regulated by law to what *extent* such loans may be granted and what *type of loans*, in particular w.r.t. maturity and amortization schedule, are allowed. The possibility of taking up supplementary mortgage loans has been subject to tight control and only allowed for certain purposes and within narrow limits and for shorter maturities. A gradual deregulation and liberalization of capital flows took place during the 1980’es, but until the de facto liberalization in 1993 it was deliberately made difficult to obtain any supplementary financing through mortgage credit institutions as long as this was not related to a turnover of the house.[4] Being a macroeconomic policy tool these restrictions have been varying over time with the conduct of macroeconomic policy.

[3] As of year 2001 this constituted a change of procedure. Repayments used to be distributed to bondholders due to a randomized selection procedure.

[4] And even in this case it was difficult until 1982. An example of a different kind of credit market policy was used in 1998, where the usual 0.3% government fee for mortgage loans was raised temporarily to 5% for loans not related to a turnover.

Given this climate of tight and changing regulation debtors have persistently perceived of mortgage financing as a rationed good, taking the *political risk* of future restrictions into account. Debtors have taken out the maximal amount of mortgage loans with as long a maturity as possible and as slow an repayment profile as possible, at least in the case of a turnover. Different variants of loans have been allowed, e.g. 20 and/or 30 year annuities and mixed type of loans with 60% annuities and 40% "serial loans".[5] At any point in time the supply of loans from the mortgage institutions has been dictated to be "the best that the law will allow". Danish mortgage loans have almost without exception been fixed rate mortgage loans, owned to a large extent domestically by life insurance companies and pension funds with a preference for long term fixed rate bonds.

One consequence of this tight regulation has been that until recently the pace of innovation in the Danish market for mortgage loans has been very slow. As a matter of fact it has been difficult until recent years to talk of much innovation at all arising from the credit institutions themselves and separate from suggestions channeled through government offices, government committees and the Central Bank. In 1976 a type of adjustable rate mortgage loans was introduced, where the outstanding debt was scheduled to be refinanced in its entirety every 5 years. These loans were funded by non-callable short term bullet bonds and formerly issued in the period 1976-1985, where a tax reform put an obstacle to this activity. However, they never gained more than a few percent of the market for newly issued loans.

Discussions about reintroducing an adjustable rate mortgage product took place in the financial sector and in government committees after 1985. The tax reform obstacle was removed in 1993, but this did not result in the introduction of new products until 1996. At this time one of the largest Danish mortgage institutions, *Realkredit Danmark A/S*[6], introduced a product with a gradual refinancing schedule allowing for a smooth reaction and transition of debtor loan rates to movements in the levels of the market interest rates. Simultaneously RD reintroduced the previously known adjustable rate loans, refinancing the remaining debt 100% at fixed intervals in time.

The motivation behind these mortgage loans were

1. to allow a flexible risk profile for the debtor w.r.t. the mix of fixed rate and adjustable rate loans, hence also w.r.t. the liquidity burden during the amortization period of the loan and
2. to allow for the funding of such loans by
   (a) issuing non-callable mortgage bonds in such a way that the strict balance principle is obeyed and
   (b) at the same time use funding instruments (bullet bonds) of a type that are internationally well known, attempting to attract more demand from foreign investors towards the Danish mortgage bond market

[5] Loans with equal repayments at each payment date.

[6] Shortened as RD.

3. to lower debtor loan rates by “ riding the yield curve". Under “normal market conditions”, with a rising 0-coupon term structure, funding and refinancing a long term loan in the short end of the market leads to lower debtor loan rates, but at the risk of rising future interest rates.

The introduction of these loans led the *European Patent Office* to grant a European wide patent to RD, cf. EPO (1999). This appears to be the first case of a mortgage loan product being patented in Europe. After numerous oppositions were made to EPO by a variety of European financial institutions, in accordance with the procedures for granted patents[7] and claiming that no patentable invention had taken place, the patent was voluntarily given up by the proprietor RD in the spring 2001. No official explanation for this act has been given; it was following both the first official responses from EPO to the oppositions and the announcement of the merger between RD and Danske Bank, the latter being part of the group of financial institutions that opposed the granted patent.

Since the introduction of these adjustable rate mortgage loans other mortgage credit institutions have offered products with similar features, and except for the public disclosure rules for patent applications part of the computational implementations behind these products are treated as commercial secrets. For this reason the present paper is kept mostly in general terms, describing some general features and a computationally simple way this general idea can be implemented. However, it should be noted that the idea of treating essential parts of the computational procedures for such loans as commercial secrets appears to be at odds with current consumer credit legislation within the European Union.[8]

This paper does not discuss models set up to describe the home owner’s choice of fixed rate loans versus variable rate loans. Investing in a house is probably the largest and most important financial decision made by many households, and utility based considerations of the optimal loan and security design for home owners is a separate topic that has recently been discussed elsewhere. Some representative examples, among others, are Campbell and Cocco (2003), Cocco (2005), Flavin and Yamashita (2002), Nielsen and Poulsen (2002) and Stanton and Wallace (1996), all including citations of numerous earlier contributions. The need for asset allocation models to include lifetime labour income, pension plans and owner occupied housing is well recognized, and it is to be expected that this part of the literature will grow and extend traditional asset allocation models. However, it is also recognized that such aspects add considerable complexity to the traditional asset allocation models.

The paper is organized as follows. In section 2 the notation as well as the mathematics of the book-keeping mechanism and balance principle is introduced. In section 3 the equations are manipulated to show a certain recursive structure facilitating closed-form expressions for the funding pattern. The possibility of negative entries in this funding pattern is a problem of practical significance; Theorem 1

[7] In addition some of the major credit institutions also filed a lawsuit in Denmark.

[8] Directive 87/102/EEC and later directives concerning credit for consumers spell out the basic philosophy and the general information requirements concerning consumer credits within the European Union.

describes conditions under which this is avoided and also some structural characteristics of the funding patterns for certain types of standard loans. Section 4 shows numerical examples of the formulas developed in section 3. In section 5 a number of sufficient conditions are given in order for the debtor's interest rate on the loan to be uniquely determined. These conditions include all "normal circumstances" and the reader can skip this section with no consequences for the understanding of the rest of the paper. Section 6 shows a general way to deal with the negative funding problem if the menu of funding instruments renders this problem unavoidable. Some numerical examples are given in section 7. A short description of possible ways to implement a gradual refinancing procedure is given in section 8. Proofs of theorems are found in the Appendix.

2 Notation and the balance principle

A loan is defined by

- The principle of the debtor's loan. Without loss of generality we only treat the case where this is normalized to 1 at the time of contracting.
- The debtor's profile of repayments $\left(Z_1^d, Z_2^d, \ldots, Z_n^d\right)$, which also defines the maturity n of the loan.
- The current interest rate y on the debtor's loan. And $-$ in case of an adjustable rate loan $-$ the rules for future adjustments of y. In cases where the value of y may change we will assume that this can only occur at payment dates. Despite the possibility of an adjustable interest rate, we will keep the notation y when no misunderstanding is possible.

We assume that the periodic payments occur at equidistant points in time and denote these payments by $P_j,\ j=1,2,\ldots,n$. The payment at time j includes

1. interest on the outstanding balance, equal to the sum $\sum_{t=j}^{n} Z_t^d$ of the remaining repayments $Z_t^d,\ t=j,\ldots,n$, and
2. the repayment Z_j^d.

The following identities are true at any date j as a matter of simple accounting:

$$\text{Debtor's payment}: \ P_j = \sum_{t=j+1}^{n} y Z_t^d + (1+y) Z_j^d \tag{1}$$

$$\text{Value (normalized)}: \ 1 = \sum_{j=1}^{n} Z_j^d \ \Leftrightarrow \ 1 = \sum_{j=1}^{n} P_j (1+y)^{-j} \tag{2}$$

The funding takes place by issuing bullet bonds of maturities $1, 2, \ldots, n$. These bullet bonds are designed such that the entire payment P_j on the portfolio of outstanding bonds matches the debtor's payment. This is the essence of the strict balance principle: Except for transaction fees and directly charged interest rate margins, the payments from the debtor are exactly matching the payments to the creditors (bondholders). The notation needed is as follows:

- The j'th funding principal is denoted by Z_j^b.
- The coupon rate on the j'th bond with principal Z_j^b is denoted by c_j.
- The market price of the j'th bond per unit face value is k_j.

In parallel to the debtor's bookkeeping, cf. equations (1)-(2), there is a similar bookkeeping for the issued bonds. The *strict balance principle* and the *funding requirement* – that the market value of the portfolio of issued bonds must match the principal of the loan – leads to the two equations:

$$\text{Bondholder's payment : } P_j = \sum_{t=j+1}^{n} c_t Z_t^b + (1+c_j) Z_j^b \tag{3}$$

$$\text{Funding requirement : } 1 = \sum_{j=1}^{n} k_j Z_j^b \tag{4}$$

The four relations, (1)-(4), can be written in matrix form as shown in (5)-(6). These equations constitute $2n+2$ equations in $3n+1$ unknowns: $\{Z_1^d, \ldots, Z_n^d\}$, $\{Z_1^b, \ldots, Z_n^b\}$, $\{P_1, \ldots, P_n\}$ and y. Given y the equations are linear, but y enters in a non-linear way. In order to proceed it is necessary to eliminate n–1 degrees of freedom.

$$\begin{bmatrix} P_1 \\ P_2 \\ \vdots \\ P_{n-1} \\ P_n \\ 1 \end{bmatrix} = \begin{bmatrix} 1+y & y & y & \ldots & y \\ 0 & 1+y & y & \ldots & y \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \ddots & \ddots & 1+y & y \\ 0 & 0 & \ldots & 0 & 1+y \\ 1 & 1 & \ldots & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} Z_1^d \\ Z_2^d \\ \vdots \\ Z_{n-1}^d \\ Z_n^d \end{bmatrix} \tag{5}$$

$$\begin{bmatrix} P_1 \\ P_2 \\ \vdots \\ P_{n-1} \\ P_n \\ 1 \end{bmatrix} = \begin{bmatrix} 1+c_1 & c_2 & c_3 & \ldots & c_n \\ 0 & 1+c_2 & c_3 & \ldots & c_n \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & 1+c_n \\ k_1 & k_2 & \ldots & k_{n-1} & k_n \end{bmatrix} \cdot \begin{bmatrix} Z_1^b \\ Z_2^b \\ \vdots \\ Z_{n-1}^b \\ Z_n^b \end{bmatrix} \tag{6}$$

For a bullet loan as well as a serial loan the profile of repayments is given in advance. For all cases, where this profile $\{Z_1^d, Z_2^d, \ldots, Z_n^d\}$ is given a priori, the bottom equation in (5) becomes redundant and we have $2n+1$ equations in only $2n+1$ unknowns: $\{Z_1^b, \ldots, Z_n^b\}$, $\{P_1, \ldots, P_n\}$ and y.

For other cases the usual practice is to specify enough structure on the profile $\{Z_1^d, Z_2^d, \ldots, Z_n^d\}$ of repayments in order to remove n–1 degrees of freedom. E.g. an annuity loan ties the profile of repayments together as a geometric series: $Z_j^d = (1+y) Z_{j-1}^d$, leaving only the size of Z_1^d to be determined. Provided the restrictions imposed lead to a unique solution, this solution must usually be found by numerical methods.[9]

Remark 1 One could also eliminate the payments P_j in the equations (5)-(6). Then the whole system could be read as $n+1$ equations in $2n+1$ unknowns: $\{Z_1^d, \ldots, Z_n^d\}$, $\{Z_1^b, \ldots, Z_n^b\}$ and y. The difference in the number of degrees of freedom compared to above is due to the fact that the system becomes homogenous:

$$\left[\begin{array}{ccccc} Z_1^d & Z_2^d & Z_3^d & \ldots & Z_n^d \\ \hline 1+y & y & y & \ldots & y \\ 0 & 1+y & y & \ldots & y \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \ddots & \ddots & 1+y & y \\ 0 & 0 & \ldots & 0 & 1+y \\ 1 & 1 & \ldots & 1 & 1 \end{array}\right] = \left[\begin{array}{ccccc} Z_1^b & Z_2^b & Z_3^b & \ldots & Z_n^b \\ \hline 1+c_1 & c_2 & c_3 & \ldots & c_n \\ 0 & 1+c_2 & c_3 & \ldots & c_n \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & 1+c_n \\ k_1 & k_2 & k_3 & \ldots & k_n \end{array}\right] \tag{7}$$

and some arbitrary magnitude constraint must be imposed. E.g. that $\sum_{j=1}^{n} Z_j^d = 1$.

These equations also cover the traditional case where the entire stream of payments from the loan are funded by issuing *one* bond as a mirror image of the loan; provided that the pricing takes place in a bond market with no arbitrage opportunities and sufficiently rich in order to determine k_j uniquely. In that case all coupon rates must be identical, i.e. $c_j \equiv c \; \forall j$, and the sequence of face values Z_j^b tells how to design that particular security as a mirror image of the loan. ▮

3 The funding pattern

After the following matrix operations in (5) as well as (6):

- subtract row 2 from row 1
- subtract row 3 from row 2
- and so forth until subtracting row n from row $n-1$

these matrix representations appear as equations (8)-(9).

[9] The question of uniqueness of the solution is addressed later on in section 5.

$$
\begin{bmatrix} P_1 - P_2 \\ P_2 - P_3 \\ \vdots \\ P_{n-1} - P_n \\ P_n \\ 1 \end{bmatrix} = \begin{bmatrix} 1+y & -1 & 0 & \dots & 0 \\ 0 & 1+y & -1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & 1+y & -1 \\ 0 & 0 & \dots & 0 & 1+y \\ 1 & 1 & \dots & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} Z_1^d \\ Z_2^d \\ \vdots \\ Z_{n-1}^d \\ Z_n^d \end{bmatrix} \tag{8}
$$

$$
\begin{bmatrix} P_1 - P_2 \\ P_2 - P_3 \\ \vdots \\ P_{n-1} - P_n \\ P_n \\ 1 \end{bmatrix} = \begin{bmatrix} 1+c_1 & -1 & 0 & \dots & 0 \\ 0 & 1+c_2 & -1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ & & & 1+c_{n-1} & -1 \\ 0 & 0 & \dots & 0 & 1+c_n \\ k_1 & k_2 & \dots & k_{n-1} & k_n \end{bmatrix} \cdot \begin{bmatrix} Z_1^b \\ Z_2^b \\ \vdots \\ Z_{n-1}^b \\ Z_n^b \end{bmatrix} \tag{9}
$$

These can be further eliminated in order to solve for the funding pattern Z_j^b and the interest rate y and also to state some general properties of the solution. Define recursively the variables m_j for $j = 1, 2, \dots, n$, setting $m_0 \equiv 0$ for notational convenience:

$$
m_1 \equiv (k_1 + m_0) \cdot (1 + c_1)^{-1}, \quad m_j \equiv (k_j + m_{j-1}) \cdot (1 + c_j)^{-1} \tag{10}
$$

and perform the following matrix operations on (8) as well as (9):

- subtract m_1 times row 1 from the bottom row, row $n+1$
- subtract m_2 times row 2 from the bottom row, row $n+1$
- and so forth until subtracting m_n times row n from the bottom row

Then we are left with the equation system (13)-(14) below, where

$$
g_j = 1 + m_{j-1} - m_j(1+y), \quad j = 1, 2, \dots, n \tag{11}
$$

Since $\sum_{j=1}^{n} Z_j^d = 1$ by normalization we can rewrite the bottom equation in (13) and obtain an equation to determine the yield y on the loan, valid in all cases:

$$
1 = \sum_{j=1}^{n} \left(m_j \cdot (1+y) - m_{j-1}\right) Z_j^d \tag{12}
$$

$$\begin{bmatrix} P_1 - P_2 \\ P_2 - P_3 \\ \vdots \\ P_{n-1} - P_n \\ P_n \\ 1 + \sum_{j=1}^{n} (m_{j-1} - m_j) P_j \end{bmatrix} = \begin{bmatrix} 1+y & -1 & 0 & \dots & 0 \\ 0 & 1+y & -1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ & & & 1+y & -1 \\ 0 & 0 & \dots & 0 & 1+y \\ g_1 & g_2 & \dots & \dots & g_n \end{bmatrix} \cdot \begin{bmatrix} Z_1^d \\ Z_2^d \\ \vdots \\ Z_{n-1}^d \\ Z_n^d \end{bmatrix} \tag{13}$$

$$\begin{bmatrix} P_1 - P_2 \\ P_2 - P_3 \\ \vdots \\ P_{n-1} - P_n \\ P_n \\ 1 + \sum_{j=1}^{n} (m_{j-1} - m_j) P_j \end{bmatrix} = \begin{bmatrix} 1+c_1 & -1 & 0 & \dots & 0 \\ 0 & 1+c_2 & -1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ & & & 1+c_{n-1} & -1 \\ 0 & 0 & \dots & 0 & 1+c_n \\ 0 & 0 & \dots & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} Z_1^b \\ Z_2^b \\ \vdots \\ Z_{n-1}^b \\ Z_n^b \end{bmatrix} \tag{14}$$

Observe that whenever the sequence of repayments $\left\{Z_1^d, Z_2^d, \dots, Z_n^d\right\}$ is independent of y, equation (12) is one linear equation in the unknown y. Whenever the pattern of repayment Z_j^d, $j = 1, 2, \dots, n$, depends on y a numerical solution is necessary. However, the interest rate y can *always* be found numerically as the solution to *one* equation without involving the pattern of funding. The coefficients m_j are sufficient; and they are readily available from market data and (10).

Remark 2 The relations above are valid for all possible payment patterns that can be constructed, hence also for zero-coupon bonds of any of the maturities $1, 2 \dots, n$. With this in mind we can rewrite the last equation in (14):

$$1 = \sum_{j=1}^{n} (m_j - m_{j-1}) P_j \tag{15}$$

Consider a unit investment in the zero-coupon bond with maturity j. If d_j denotes the price of such a zero-coupon bond, also termed the *zero-coupon discount factor* for time j, such a unit investment buys a principal of zero coupon bonds equal to the inverse $1/d_j$ of this discount factor. Hence, $(m_j – m_{j-1}) = d_j$.

These zero-coupon discount factors are positive numbers in a bond market void of arbitrage opportunities. In a market operating under “normal conditions” the *forward rates* are also positive. Hence, under “normal conditions” the magnitudes $(m_j - m_{j-1})$ are decreasing with j. ▮

For *illustrative purposes only* consider first the special case where all funding bonds have the same coupon rate c and the term structure is flat at level y. Although not a realistic case it shows that the magnitudes derived have natural interpretations. Under these circumstances the bullet bond prices are

$$k_j = \frac{c}{y} + \left(1 - \frac{c}{y}\right) (1+y)^{-j} \tag{16}$$

and by induction it can be proved that m_j is equal to the annuity factor:

$$m_j = \alpha_{\overline{j}|y} \equiv \frac{1-(1+y)^{-j}}{y} \tag{17}$$

Hence

$$m_j(1+y) - m_{j-1} = 1, \quad g_j = 0, \quad m_{j-1} - m_j = -d_j = -(1+y)^{-j} \tag{18}$$

The latter equation in (14) is then identical to the latter equation in (13), expressing the present value requirement

$$1 = \sum_{j=1}^{n} P_j(1+y)^{-j}$$

whereas equation (12) simply reproduces the normalized revenue requirement.

In the *general* case the magnitudes m_j appear as the solution to the forward difference equation (10):

$$m_j = \sum_{t=1}^{j} \frac{h_j}{h_t}(1+c_t)^{-1}k_t \tag{19}$$

where the terms h_j are defined by

$$h_j \equiv \prod_{t=1}^{j}(1+c_t)^{-1} \tag{20}$$

The funding principals are then found from the backward difference equation (14) with the solution

$$Z_j^b = \sum_{t=j}^{n} \frac{h_t}{h_j}(1+c_j)^{-1}\left[(1+y)Z_t^d - Z_{t+1}^d\right] = \sum_{t=j}^{n} \frac{h_t}{h_j}(1+c_j)^{-1}\left[P_t - P_{t+1}\right] \tag{21}$$

Variables with an index beyond their domain of definition are defined to have the value zero.

According to the regulatory rules it is not allowed to have any of the funding principals Z_j^b negative. In practice this would imply that the mortgage credit institution should issue "too many" bonds of some maturities in order to raise funds beyond the need of the debtor. These surplus funds should then be used to buy back bonds at the maturities, where the mathematical solution returns a negative Z_j^b – called "negative funding". Nothing in the – essentially linear – mathematics prevents this from occurring in general, but a solution must be found in order to correct this whenever it occurs. In some cases the problem will disappear by lowering the coupon rates, but this is not always possible.[10]

With this background of notation and mathematical formulation we state the first theorem.

Theorem 1 *For any loan funded by issuing bullet bonds in accordance with the strict balance principle the following is true:*

1. *The funding principal at the longest maturity, Z_n^b, is always positive.*

[10]Tax law considerations and the asymmetric taxation of interest payments and capital gains has lead to an implicit lower limit on the coupon rates that is used.

2. *If the payments $P_1, P_2, \ldots, P_n$ are non-increasing, negative funding will never occur.*

3. *If the repayments Z^d_j are increasing between all payments dates with at most the factor $1+y$, negative funding will never occur.*

4. *If the payments $P_1, P_2, \ldots, P_n$ are non-decreasing, negative funding may occur. If so, the sequence of funding principals $\{Z^b_1, Z^b_2, \ldots, Z^b_n\}$ will have exactly one sign change.*

5. *For annuity loans, the funding principals satisfy*

$$Z^b_j = \frac{d_n}{d_j} Z^b_n, \qquad Z^b_n = \frac{1+y}{1+c_n} Z^d_n = (1+c_n)^{-1} \alpha^{-1}_{\overline{n}|y} \tag{22}$$

6. *If the payments $P_1, P_2, \ldots, P_{n-1}$ are non-increasing, negative funding may occur due to the last payment P_n. If so, the sequence of funding principals $\{Z^b_1, Z^b_2, \ldots, Z^b_{n-1}\}$ will have exactly one sign change.*

7. *For bullet loans, the funding principals $\{Z^b_1, Z^b_2, \ldots, Z^b_{n-1}\}$ are all of the same sign. The sign is determined solely by the sign of $y - c_n$.*

   For annuity loans with payments in accordance with a maturity m, but where the outstanding principal is prepaid prematurely as a balloon payment at time n, $n < m$, all the funding principals $\{Z^b_1, Z^b_2, \ldots, Z^b_{n-1}\}$ are also all of the same sign. This sign is determined by the sign of $1 - c_n \alpha_{\overline{m-n}|y}$.

▮

Proof

1. $Z^b_n = \dfrac{1+y}{1+c_n} Z^d_n$.

2. *This is a direct consequence of the second equality in (21).*

3. *This is a direct consequence of the first equality in (21).*

4. *This also follows from (21). If negative funding ever occurs it can only happen after sufficiently many negative elements of the form $P_t - P_{t+1}$ in the backwards working solution (21). And once observed all preceding funding principals must be negative as well.*

5. *For annuities the repayments form a geometric series: $Z^d_{t+1} = (1+y) Z^d_t$. Hence only the last term in the first equality in (21) contribute. Alternatively, since all payments are equal only the last term in the second equality of (21) contribute.*

   As $P_n = \alpha^{-1}_{\overline{n}|y} = (1+y) Z^d_n = (1+c_n) Z^b_n$ the relation (22) follows.

6. *This also follows from (21). If negative funding $Z^b_{n-1} < 0$ occurs because of the last payment this will continue backwards until corrected by sufficiently many positive elements of the form $P_t - P_{t+1}$. And once observed all preceding funding principals must be positive as well.*

7. *For a bullet loan,* $Z_n^d = 1$ *and* $Z_{n-1}^d = Z_{n-2}^d = \ldots = Z_1 = 0$. *Only two terms – the last two terms – in (21) determine the sign of the funding principals* $Z_{n-1}^b, Z_{n-2}^b, \ldots, Z_1^b$. *We have*

$$Z_{n-1}^b = (1+c_n)^{-1}(1+c_{n-1})^{-1}(1+y) - (1+c_{n-1})^{-1} = \frac{y - c_n}{(1+c_{n-1})(1+c_n)} \tag{23}$$

$$Z_j^b = (1+c_j)^{-1} Z_{j+1}^b, \quad j = n-2, n-3, \ldots, 1 \tag{24}$$

For the annuity loan with balloon payment we have that

$$Z_n^d = \alpha_{\overline{m}|y}^{-1} \alpha_{\overline{m+1-n}|y} \tag{25}$$

$$(1+y)Z_{n-1}^d - Z_n^d = \alpha_{\overline{m}|y}^{-1} \left[(1+y)^{-(m+1-n)} - \alpha_{\overline{m+1-n}|y} \right] = -\alpha_{\overline{m}|y}^{-1} \alpha_{\overline{m-n}|y} \tag{26}$$

$$\begin{aligned} Z_{n-1}^b &= (1+c_{n-1})^{-1} \alpha_{\overline{m}|y}^{-1} \left[(1+c_n)^{-1}(1+y)\alpha_{\overline{m+1-n}|y} - \alpha_{\overline{m-n}|y} \right] \\ &= (1+c_n)^{-1}(1+c_{n-1})^{-1} \alpha_{\overline{m}|y}^{-1} \left[1 - c_n \alpha_{\overline{m-n}|y} \right] \end{aligned} \tag{27}$$

$$Z_j^b = (1+c_j)^{-1} Z_{j+1}^b, \quad j = n-2, n-3, \ldots, 1 \tag{28}$$

$\blacksquare$

4 Examples

In Tables 1 and 2 we give four examples, an annuity loan and a bullet loan under two different term structures. Both term structures are typically rising 0-coupon term structures, and the funding instruments have been chosen to have the same coupon rate. This can easily be changed. Coupon rates, yields and borrowing rates are in % p.a. The discount factors $m_j - m_{j-1}$ are denoted as d_j.

Examples 1−2

t	coupon rate	k_j	d_j	0-coupon yield	$100 \cdot Z_j^b$ (annuity)	$100 \cdot Z_j^b$ (bullet)
1	4,00	1,01	0,9712	2,9703	8,5784	0,6005
2	4,00	1,01	0,9338	3,4839	8,9215	0,6245
	4,00	1,00	0,8883	4,0283	9,2784	0,6495
4	4,00	0,99	0,8445	4,3161	9,6495	0,6755
5	4,00	0,98	0,8024	4,5014	10,0355	0,7025
6	4,00	0,97	0,7619	4,6362	10,4369	0,7306
7	4,00	0,96	0,7230	4,7426	10,8544	0,7598
8	4,00	0,95	0,6856	4,8318	11,2886	0,7902
9	4,00	0,94	0,6496	4,9102	11,7401	0,8218
10	4,00	0,93	0,6150	4,9816	12,2097	100,8547
Borrowing rate y					4,5968	4,8889

Table 1: Funding principals with no negative funding

Examples 3−4

t	coupon rate	k_j	d_j	0-coupon yield	$100 \cdot Z_j^b$ (annuity)	$100 \cdot Z_j^b$ (bullet)
1	6,00	1,02	0,9623	3,9216	7,2460	-0,2898
2	6,00	1,03	0,9172	4,4145	7,6808	-0,3072
3	6,00	1,04	0,8747	4,5617	8,1416	-0,3257
4	6,00	1,05	0,8347	4,6217	8,6301	-0,3452
5	6,00	1,06	0,7969	4,6464	9,1479	-0,3659
6	6,00	1,06	0,7517	4,8708	9,6968	-0,3879
7	6,00	1,06	0,7092	5,0314	10,2786	-0,4114
8	6,00	1,05	0,6596	5,3388	10,8953	-0,4358
9	6,00	1,05	0,6223	5,4120	11,5490	-0,4620
10	6,00	1,04	0,5776	5,6417	12,2420	99,5103
Borrowing rate y					5,0414	5,4809

Table 2: Funding principals with negative funding

In the first two examples, cf. Table 1, the long term bond is priced below par – hence there are no problems with negative funding in the bullet loan, cf. theorem 1, items (1) and (7). In the other two examples, cf. Table 2, the bonds are all priced above par, in particular the bond with the longest maturity. The annuity loan does not cause any problems in this respect, whereas the bullet loan will have negative funding principals at all maturities shorter than the maximal maturity 10 in accordance with item (7) of theorem 1.

It is apparent that debtor's interest rate y in all four examples is dominated by the 0-coupon rate in the long end. In Table 1 the spread between the one year and the ten year 0-coupon rate is approximately 2%. The bullet loan has an interest rate only 10 bp below the 10-year 0-coupon rate, whereas the annuity has an interest rate 39 bp below the 10-year 0-coupon rate. In Table 2 the spread between the one year and the ten year 0-coupon rate is 172 bp. The bullet loan has an interest rate only 16 bp below the 10-year 0-coupon rate, whereas the annuity has an interest rate 60 bp below the 10-year 0-coupon rate.

This reflects the well know fact that the interest rate on a portfolio is closely approximated by a duration-weighted average of the individual yields.

5 Uniqueness of the interest rate y

It remains to be examined under what circumstances a *unique* interest rate y can be determined from the equations (5)-(6) or, equivalently, (8)-(9).[11]

[11] As will be apparent from theorem 2 below this is not a real issue under any set of realistic assumptions. Hence, this section can be omitted without lack of continuity.

Since the funding principals are given in (21) the funding requirement can be directly stated $-$ in alternative versions $-$ as

$$\begin{aligned}
1 &= \sum_{j=1}^{n} k_j Z_j^b \\
&= \sum_{j=1}^{n} k_j \sum_{t=j}^{n} \frac{h_t}{h_j}(1+c_j)^{-1}\left[(1+y)Z_t^d - Z_{t+1}^d\right] \\
&= \sum_{t=1}^{n} \left[(1+y)Z_t^d - Z_{t+1}^d\right] \sum_{j=1}^{t} k_j \frac{h_t}{h_j}(1+c_j)^{-1} \\
&= \sum_{t=1}^{n} \left[(1+y)Z_t^d - Z_{t+1}^d\right] m_t \qquad (29) \\
&= \sum_{t=1}^{n} \left[P_t - P_{t+1}\right] m_t \qquad (30) \\
&= \sum_{t=1}^{n} P_t \left[m_t - m_{t-1}\right] \qquad (31)
\end{aligned}$$

A sufficient, but not necessary, condition in order to guarantee uniqueness is that

$$\frac{\partial}{\partial y}\left[(1+y)Z_t^d - Z_{t+1}^d\right] = \frac{\partial}{\partial y}\left[P_t - P_{t+1}\right] \qquad (32)$$

has the same sign[12] over the entire domain of definition of y, i.e. $(-1,\infty)$. Since $Z_{n+1}^d = P_{n+1} \equiv 0$ this sign should also be the sign of $\partial P_n^d/\partial y = \partial[(1+y)Z_n^d]/\partial y$.

This condition is valid in many cases of practical significance. It is trivially the case in *all* scenarios where the profile of repayments is independent of y, e.g. bullet loans and serial loans with equal repayments of principal at every payment.[13] It is also the case for annuities, where all the terms $[P_t - P_{t+1}]$ are zero except for the very last one, which is equal to the annuity payment:

$$P_n = (1+y)Z_n^d = \alpha_{\overline{n}|y}^{-1}$$

which is clearly an increasing function of y.

This observation carries over to portfolios of loans satisfying these sufficiency conditions. E.g. a mixture of an annuity, a bullet loan and a serial loan satisfies the sufficiency condition in (32).[14]

As pointed out in relation to (15) the coefficients $m_t - m_{t-1}$ are the implicit discount factors that can be derived from the bond market. If there are no arbitrage opportunities in the bond market these coefficients are:

[12] Zeroes can be ignored.

[13] As noted above the interest rate y can be found under these circumstances as the solution of one linear equation in one unknown.

[14] A mixture of 60% annuity and 40% serial loan were required by law in the period 1986-1993 for most Danish mortgage loans.

1. uniquely determined positive numbers
2. independent of the coupon rates on the underlying bullet bonds
3. under "normal assumptions", reflecting positive forward rates, these coefficients are also decreasing numbers in the interval (0,1)

Under the "no arbitrage" assumption it is sufficient that $\partial P_t/\partial y$ is of the same sign $\forall t$. If, furthermore, there are no negative forward rates, it is sufficient that all partial sums $\sum_{t=1}^{j} \partial P_t/\partial y$, $j = 1, 2, \ldots, n$, have the same sign.

Summarizing these rather general characterizations of sufficient conditions we have.

Theorem 2 *In order for the interest rate y on the loan to be uniquely determined it is sufficient that one of the following conditions are satisfied.*

1. *The profile of repayments Z_j^d, $j = 1, 2, \ldots, n$, is independent of y.*
2. *The derivatives*
$$\frac{\partial}{\partial y}\left[(1+y)Z_t^d - Z_{t+1}^d\right] = \frac{\partial}{\partial y}\left[P_t - P_{t+1}\right]$$
have the same sign for $t = 1, 2, \ldots, n$. Since $Z_{n+1} = P_{n+1} \equiv 0$, this sign must be the same as the sign of $\partial P_n^d/\partial y$.
3. *Any portfolio of loans individually satisfying either condition (1) or condition (2) with identical signs across the individual loans in the portfolio.*
4. *If the bond market is arbitrage free it is sufficient that the derivatives $\partial P_t/\partial y$ have the same sign for $t = 1, 2, \ldots, n$. This is in particular satisfied for all payment profiles of the form*
$$P_{t+1} = aP_t, \quad P_n = a^{n-1}P_1 + \sum_{t=n+1}^{m} a^{t-1}P_1(1+y)^{-(t-n)} \tag{33}$$
where $a > 0$ and P_1 is set in order to fulfill the present value condition.
5. *If – in addition to being arbitrage free – the bond market is void of negative forward rates it is sufficient that all partial sums $\sum_{t=1}^{j} \partial P_t/\partial y$, $j = 1, 2, \ldots, n$ have the same sign.*

Proof

1. *If the profile of repayments Z_t^d is independent of y, equation (29) is one linear equation in y.*
2. *If this sign condition is satisfied, the rhs of (29) is a monotone function of y, because all coefficients m_t are positive.*
3. *Trivial.*

4. *The proof of this is devoted to the Appendix.*
5. *The proof of this is devoted to the Appendix.* ∎

Remark 3 The payment profiles in (33) cover loans of a certain systematic type.[15] For $a = 1$ it is an annuity with periodic payments calculated relative to an “as if” maturity of m, but with a prescribed balloon payment equal to the remaining debt at time n. Letting $m \to \infty$ gives the bullet loan as the limiting case. ∎

The possibility of constructing payment profiles with such a degree of interaction between the interest rate and the payment profile that problems of non-uniqueness of the interest rate y show up is a highly theoretical one in view of theorem 2 and non-existing for any practical purpose.

6 Negative funding

The origin of negative funding is the possibility of increasing payments, cf. theorem 1, items (4)-(6). This occurs most outspokenly for a bullet loan, but will also occur for other loans with a final payment that changes the nature of the payment profile $\{P_1, P_2, \ldots, P_{n-1}\}$.

Consider first the case of a bullet loan, where we know that all funding principals Z_j^b, $j = 1, 2, \ldots, n-1$, are of the same sign. If they are all negative the following equation solves this problem by “rolling back” the negative funding. Since an increase in a negative funding variable Z_j^b must necessarily be compensated by a decrease in another funding variable there is only one way this can be accomplished for a bullet loan: All funding must be placed at the longest maturity bond Z_n^b.

As will become clear below this will in itself correct the negative funding problem. However, the payment profile arising from this recalculation does *not* reflect a true bullet loan, although it resembles closely the bullet loan. It has some rather small intermediate repayments and a large balloon payment at maturity.

In order to preserve the funding requirement the changes in funding must obey the relation

$$\sum_{j=1}^{n-1} k_j(-Z_j^b) + k_n \Delta Z_n^b = 0 \quad \Rightarrow \quad \Delta Z_n^b = \frac{1}{k_n} \sum_{j=1}^{n-1} k_j Z_j^b \tag{34}$$

Furthermore, from (13) and (14) it is apparent that the revised payment profile for the first $n-1$ payments forms an annuity, just like the payments in a bullet loan. Debtor’s interest rate y, the internal rate of return on the sequence of payments, is the yield on the single funding instrument. In an increasing 0-coupon term structure this will lower the true interest rate below the interest rate found for the payments based on negative funding.

[15]The loans described belong to the class of socalled “systematic loans” discussed at length in Hasager and Jensen (1990).

Example 5

Consider the previous examples in Table 2. Performing the recalculation in (34) leads to the payments in Table 3.

The *true* bullet loan can be re-established provided the small repayments are planned to be neutralized by future funding, selling more of the n-maturity bond in time with the repayments to be neutralized.[16] Since future market conditions cannot be known at the date of contracting this involves changes in the interest rate y in time with every new funding issue. In the example additional funding of 0,2991 is necessary at time 1, lowering the debtor payment from 5,7692 to 5,4701, which is the interest payment on the principal 100. After this additional funding the rest of the planned new funding (the remaining entries in the column "Z_j^d corrected") are no longer valid unless the interest rate y remains unchanged by coincidence. In other words it is necessary to give up either the bullet loan feature or the fixed rate feature.

t	coupon rate	k_j	ΔP_j	P_j (corrected)	P_j (original)	Z_j^d (corrected)	$100 \cdot Z_j^b$ (corrected)	$100 \cdot Z_j^b$ (bullet)
1	6,00	1,02	0,2883	5,7692	5,4809	0,2991	0	-0,2898
2	6,00	1,03	0,2883	5,7692	5,4809	0,3155	0	-0,3072
3	6,00	1,04	0,2883	5,7692	5,4809	0,3328	0	-0,3257
4	6,00	1,05	0,2883	5,7692	5,4809	0,3510	0	-0,3452
5	6,00	1,06	0,2883	5,7692	5,4809	0,3702	0	-0,3659
6	6,00	1,06	0,2883	5,7692	5,4809	0,3904	0	-0,3879
7	6,00	1,06	0,2883	5,7692	5,4809	0,4118	0	-0,4114
8	6,00	1,05	0,2883	5,7692	5,4809	0,4343	0	-0,4358
9	6,00	1,05	0,2883	5,7692	5,4809	0,4580	0	-0,4620
10	6,00	1,04	-3,5578	101,9231	105,4809	96,6369	96,1538	99,5103
Borrowing rate							5,4701	5,4809

Table 3: Funding principals for a modified bullet loan

∎

The suggested procedure shown in Table 3 is a feasible correction to the negative funding problem. It is also optimal in the sense that it minimizes the total amount of additional funding. More generally, define the supplementary funding values by the non-negative variables $\widetilde{Z}_j^d$, $j = 1, 2, \ldots, n-1$. As seen above they can be thought of as "negative repayments", hence a supplementary funding is a reduction in the planned repayment Z_j^d that would arise from a funding procedure where all the funding is issued at the beginning for the period. Minimizing the total amount of "negative repayment" can be formulated as a

[16] The additional funding does not have to carry the same coupon rate as the old n-maturity bond.

quasi-linear programming problem. In mathematical terms and for a given value of y this is written in terms of a standard simplex tableau in (35) below.

If it is necessary to take out supplementary funding at time 1 the payments on the funding instruments exceed the desired payments on the loan. The difference is exactly $\widetilde{Z}_1^d$, reflecting an additional loan to be taken up and funded by funding instruments and on market terms valid at time 1. This is the coefficient "-1" in the first row.

Interest must be paid on this additional loan at times $t=2,3,\ldots,n{-}1$. This is reflected in the coefficients y in the column under $\widetilde{Z}_1^d$ at entries $t=2,3,\ldots,n-1$. At time n the loan must be repaid, reflected in the coefficient $1{+}y$ at entry n. Hence, for a given and guessed value of y the objective function is to minimize the total amount of additional funding, subject to the constraints that in any period the payments

- on the funding portfolio
- plus the interest payments on already issued additional funding
- less the possible additional funding issued in that period

must match the planned payments. In case a non-negative funding scheme exists, the values of all the additional funding variables $\widetilde{Z}_j^d$, $j=1,2,\ldots,n-1$, will be driven to zero.

$$\text{Minimize } \sum_{j=1}^{n-1} \widetilde{Z}_j^d$$
$$\{\widetilde{Z}_j^d \geq 0, Z_j^b \geq 0, y\}$$
$$\text{subject to}$$

$$\begin{array}{|cccccccccc|}
\hline
Z_1^b & Z_2^b & Z_3^b & \ldots & Z_n^b & \widetilde{Z}_1^d & \widetilde{Z}_2^d & \widetilde{Z}_3^d & \ldots & \widetilde{Z}_{n-1}^d \\
\hline
0 & 0 & 0 & \ldots & 0 & -1 & -1 & -1 & \ldots & -1 \\
\hline
1+c_1 & c_2 & c_3 & \ldots & c_n & -1 & 0 & 0 & \ldots & 0 \\
0 & 1+c_2 & c_3 & \ldots & c_n & y & -1 & 0 & \ldots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ldots & \ddots \\
0 & \ddots & \ddots & 1+c_{n-1} & c_n & y & y & \ddots & & -1 \\
0 & 0 & \ldots & 0 & 1+c_n & 1+y & 1+y & 1+y & \ldots & 1+y \\
\hline
\end{array} =$$

$$\begin{array}{|ccccc|}
\hline
Z_1^d & Z_2^d & Z_3^d & \ldots & Z_n^d \\
\hline
0 & 0 & 0 & \ldots & 0 \\
\hline
1+y & y & y & \ldots & y \\
0 & 1+y & y & \ldots & y \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \ldots & \ldots & 1+y & y \\
0 & 0 & \ldots & \ldots & 1+y \\
\hline
\end{array} \qquad (35)$$

Only the interest rate y enters in a non-linear fashion. For a given y the optimal value of the funding variables Z_j^b, $j = 1, 2, \ldots, n$ can be checked to see if the funding requirement

$$\sum_{j=1}^{n} k_j Z_j^b = 1 \tag{36}$$

If the solution found for Z_j^b falls short of the funding requirement, a marginal increase in y can be made and the solution can be recalculated. A search procedure can be continued until the funding requirement is matched within the required precision.

By standard row operations we can reformulate (35) as

$$\begin{gathered} \text{Minimize } \sum_{j=1}^{n-1} \widetilde{Z}_j^d \\ \{\widetilde{Z}_j^d \geq 0, Z_j^b \geq 0, y\} \\ \text{subject to} \end{gathered}$$

$$\begin{array}{|cccccccccc|}
\hline
Z_1^b & Z_2^b & Z_3^b & \ldots & Z_n^b & \widetilde{Z}_1^d & \widetilde{Z}_2^d & \widetilde{Z}_3^d & \ldots & \widetilde{Z}_{n-1}^d \\
\hline
0 & 0 & 0 & \ldots & 0 & -1 & -1 & -1 & \ldots & -1 \\
\hline
1+c_1 & -1 & 0 & \ldots & 0 & -1-y & 1 & 0 & \ldots & 0 \\
0 & 1+c_2 & -1 & \ldots & 0 & 0 & -1-y & 1 & \ldots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ldots & \ddots \\
0 & \ddots & \ddots & 1+c_{n-1} & -1 & -1 & -1 & \ddots & & -(2+y) \\
0 & \ddots & \ddots & 0 & 1+c_n & 1+y & 1+y & 1+y & \ldots & 1+y \\
\hline
\end{array} =$$

$$\begin{array}{|ccccc|}
\hline
Z_1^d & Z_2^d & Z_3^d & \ldots & Z_n^d \\
\hline
0 & 0 & 0 & \ldots & 0 \\
\hline
1+y & -1 & 0 & \ldots & 0 \\
0 & 1+y & -1 & \ldots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \ldots & \ldots & 1+y & -1 \\
0 & 0 & \ldots & \ldots & 1+y \\
\hline
\end{array} \tag{37}$$

The problem can also be solved as an optimization problem in all $2n$ variables:

$$Z_1^b, Z_2^b, \ldots, Z_n^b, \widetilde{Z}_1^d, \widetilde{Z}_2^d, \ldots, \widetilde{Z}_{n-1}^d, y$$

simultaneously with the constraints (36)-(37) and the non-negativity conditions. Numerous numerical procedures exist for this type of constrained optimization.

The same pattern repeats itself at time $2, 3, \ldots, n-1$. However, the existing payments from already issued funding instruments must be taken to the r.h.s. and only the additional funding must be found.

When new funding is issued at time under market conditions unknown at time 0, a re-calculation of the interest rate on the loan is necessary in order to equate the present value of the planned payments with the remaining balance on the loan.

From the bottom equation of (37) it is apparent that the objective of minimizing the amount of additional funding is equivalent to maximizing the funding variable Z_n^b, subject to the constraints in (37). Hence, if a feasible solution can be found where funding *only* takes place by issuing the bond of longest maturity it is also optimal.

Bullet loans and annuity loans with a balloon payment share the property that the funding can be carried out by issuing only the longest term bond. Hence, from the upper $n-2$ equations in (37) the additional funding follows a geometric series: $\widetilde{Z}_{j+1}^d = (1+y)\widetilde{Z}_j^d$. Given this, each of the upper $n-2$ equations reduce to the empty statement "$0=0$".

Consider first the case of a bullet loan. Solving the last two equations in (37) backwards $-$ knowing that $Z_n^b = 1/k_n$ $-$ gives the following result:

$$\frac{1+c_n}{k_n} = (1+y)\left[1 - \sum_{j=1}^{n-1} \widetilde{Z}_j^d\right] \tag{38}$$

$$\left[1 - \sum_{j=1}^{n-1} \widetilde{Z}_j^d\right] = (1+y)\widetilde{Z}_{n-1}^d + \frac{1}{k_n} \tag{39}$$

Theorem 3 *The equations (38)-(39) have a unique solution for the interest rate* $y \in (-1, \infty)$ *and the sequence* $\widetilde{Z}_j^1, \ldots, \widetilde{Z}_{n-1}^d$. *This is also the case for an annuity loan with a balloon payment and the corresponding equations for that case.* ▮

Proof *The proof of the uniqueness is purely technical and devoted to the Appendix. Similarly, the necessary modifications for the annuity loan with a balloon payment is found in the Appendix.* ▮

Example 5 (continued)

Consider Table 3 and the first re-funding operation that takes place at time 1. The planned payments on the loan are 5.4701 for the first 9 periods and 105.4701 at maturity. At time 1 additional funding of 0.2991 is raised in order to meet the coupon payment 5.7692 on the issued 10-year bonds.

The optimization problem related to the additional funding at time 1 requires the market conditions at time 1 as input. Assume for the purpose of illustration that market rates have dropped and that the market conditions are in accordance with the data shown in Table 1. The optimal solution only involves use of the longest bond, now with maturity 9 years, carrying 4% coupon and sold at 94% of face value and with the yield 4.8381%. After having performed the same row operations as those leading to (37) the optimal solution for *new* funding Z_{10}^b and *new* planned additional funding involves solving the equations (40)-(42), where $\widetilde{Z}_{t+1}^d = (1+y) \cdot \widetilde{Z}_t^d$ for $t = 2, 3, \ldots, 8$:

$$Z_{10}^b + \sum_{t=2}^{8} \widetilde{Z}_t^d + (2+y) \cdot \widetilde{Z}_9^d = 100 - \frac{100}{1.04} \tag{40}$$

$$1.04 \cdot Z_{10}^b + (1+y) \cdot \sum_{t=2}^{9} \widetilde{Z}_t^d = (1+y) \cdot 100 - \frac{106}{1.04} \tag{41}$$

$$0.94 \cdot Z_{10}^b = 0.2991 \tag{42}$$

The solution is

$$Z_{10}^b = 0.3182 \qquad \widetilde{Z}_2^d = 0.3138 \qquad y = 5.4681 \tag{43}$$

The large drop in interest rates only lowers the borrowing rate from 5.4701% to 5.4681%. The need for additional funding at time 2 has diminished marginally from 0.3155 to 0.3138. ▮

As an example where the sole use of the longest term bond is not feasible consider the case of a serial loan with a balloon payment. For such a loan we know from theorem 1, item (6), that the negative funding occurs - if at all - in a sequence like $j, j+1, \ldots, n-1$.

Example 6

We take the following input data for a four period maturity horizon and an underlying 40 period serial loan with balloon payment at maturity 4 for granted.

t	coupon (% p.a.)	k_t	0-coupon rate	yield
1	8	1.04	3.8462	3.8462
2	8	1.08	3.7694	3.7723
3	8	1.11	4.0523	4.0336
4	8	1.12	4.7318	4.6438

Table 4: Input data for a serial loan with negative funding

Z_1^b	Z_2^b	Z_3^b	Z_4^b	$\widetilde{Z}_1^d$	$\widetilde{Z}_2^d$	$\widetilde{Z}_3^d$	y
32.29	-1126.53	-2378.06	896270.29	0	0	0	4.6456
9.49	0	0	892848.33	0	1150.70	2365.09	4.6438

Table 5: Funding pattern for a serial loan with negative funding

Table 5 shows the unrestricted funding pattern with negative funding at maturities 2 and 3 − in accordance with theorem 1, items (4) and (6) − together with the optimal restricted funding pattern for a loan with principal 1.000.000.[17]

As can be read of from the reduced form expression of the funding equations in (37), any correction of a funding variable, say Z_j^b, from negative to zero − keeping the interest rate y unchanged − induces a need for all previous funding variables Z_{j-i} to increase. The necessary changes in y are usually very small, so this property is expected to show up in the end result. This is also as shown in the example above, where Z_3^b is set at zero, Z_2^b is increased to a positive level and Z_1^b is increased above the level found in the unrestricted solution. The effect on y, however, is only 24 bp. ▮

Example 7

Analogous to example 6 we take the following input data for a four period maturity horizon and an underlying 40 period annuity loan with balloon payment at maturity 4:

t	coupon (% p.a.)	k_t	0-coupon rate	yield
1	6	1.04	1.9231	1.9231
2	6	1.07	2.3879	2.3752
3	6	1.10	2.5145	2.4987
4	6	1.12	2.8255	2.7880

Table 6: Input data for an annuity loan with negative funding

Table 7 shows the unrestricted funding pattern with negative funding at all maturities except the longest one − in accordance with theorem 1, item (7) − together with the optimal restricted funding pattern for a loan with principal 1.000.000:

Z_1^b	Z_2^b	Z_3^b	Z_4^b	$\widetilde{Z}_1^d$	$\widetilde{Z}_2^d$	$\widetilde{Z}_3^d$	y
-11622.14	-12319.47	-13058.64	928244.07	0	0	0	2.7967
0	0	0	892857.14	11778.93	12107.33	12444.89	2.7880

Table 7: Funding pattern for an annuity loan with negative funding

▮

[17] The calculations have been performed in MATLAB with a very high degree of precision. Rounding off the interest rate to 4 decimals as reported in Table 5 is an insufficient degree of precision to redo the amortization; in practice these calculations are performed with 8 decimals precision by the mortgage credit institutions.

7 Fully refinanced mortgage loans

A fully refinanced mortgage loan is a mortgage loan, where the payment schedule is set on an "as if" basis. I.e. the payments are calculated as if the loan was to be amortized over a certain time horizon m according to some fixed schedule. It is agreed that the remaining debt is subject to refinancing every n years. A 30 year loan may be refinanced fully every 2,3,5,6,10 or 15 years with all adjustment periods of equal length. However, re-financing may occur every 4 years. In that case at least one of the adjustment periods must be different from 4 years.[18]

The objective embodied in such a mortgage loan is two-fold. One objective is to have a degree of adjustability in the long term mortgage rate instead of basing the conditions for maybe a 30 year mortgage upon specific market conditions at a particular date. Another objective is, of course, that the debtor under conditions of rising term structures can "ride the yield curve" by funding a long term borrowing need in the short end of the bond market. This involves, of course, the risk of future rising yield curves.

A compromise between these two objectives involves a choice on behalf of the debtor. The highest degree of "yield curve riding" is the "F1-loan", where the entire outstanding debt is refinanced every year by issuing new bonds with one year to maturity. This will ensure the debtor the lowest payments in the short run, but it will also give the highest degree of variability in the debtor's payments in the long run.

We can illustrate the procedure by using the input data from our previous examples, cf. Table 1 and Table 2.

t	coupon rate	k_j	h_j	m_j	d_j	0-coupon rate	$100 \cdot Z_j^b$ (annuity)	$100 \cdot Z_j^b$ (bullet)
1	4,00	1,01	0,9615	0,9712	0,9712	2,9703	8,4832	0,3702
2	4,00	1,01	0,9246	1,9050	0,9338	3,4839	8,8226	0,3850
3	4,00	1,00	0,8890	2,7932	0,8883	4,0283	9,1755	0,4004
4	4,00	0,99	0,8548	3,6377	0,8445	4,3161	9,5425	0,4165
5	4,00	0,98	0,8219	4,4401	0,8024	4,5014	65,2027	100,4331
Borrowing rate							4,3297	4,4504

Table 8: Funding principals with no negative funding. $m = 10$, $n = 5$.

t	coupon rate	k_j	h_j	m_j	d_j	0-coupon rate	$100 \cdot Z_j^b$ (annuity)	$100 \cdot Z_j^b$ (bullet)
1	4,00	1,01	0,9615	0,9712	0,9712	2,9703	8,0347	-0,4853
2	4,00	1,01	0,9246	1,9050	0,9338	3,4839	90,9752	99,4952
Borrowing rate							3,4521	3,4751

[18] These loans are termed "F-loan", and with re-financing every n years the loans are characterized as "Fn-loans".

Table 9: Funding principals with negative funding. $m=10$, $n=2$.

t	coupon rate	k_j	h_j	m_j	d_j	0-coupon rate	$100 \cdot Z_j^b$ (annuity)	$100 \cdot Z_j^b$ (bullet)
1	6,00	1,02	0,9434	0,9623	0,9623	3,9216	6.9971	-1.0223
2	6,00	1,03	0,8900	1,8795	0,9172	4,4145	7.4170	-1.0836
3	6,00	1,04	0,8396	2,7542	0,8747	4,5617	7.8620	-1.1487
4	6,00	1,05	0,7921	3,5889	0,8347	4,6217	8.3337	-1.2176
5	6,00	1,06	0,7473	4,3858	0,7969	4,6464	64.4308	98.7094
Borrowing rate							4,5991	4,6319

Table 10: Funding principals with negative funding. $m=10$, $n=5$

t	coupon rate	k_j	h_j	m_j	d_j	0-coupon rate	$100 \cdot Z_j^b$ (annuity)	$100 \cdot Z_j^b$ (bullet)
1	6,00	1,02	0,9434	0,9623	0,9623	3,9216	7,1181	-1,4206
2	6,00	1,03	0,8900	1,8795	0,9172	4,4145	90,0384	98,4942
Borrowing rate							4,0781	4,4038

Table 11: Funding principals with negative funding. $m=10$, $n=2$

8 Partially refinanced loans

Instead of loans where the outstanding debt is scheduled for a full refinancing at predetermined points in time the idea of a gradual refinancing has also been introduced. Behind this idea is, of course, again the attempt to “ride the yield curve” in the sense that a long term borrowing need is funded sequentially by bonds with much shorter maturity and – under “normal” conditions in the bond market – with lower yields. The change in the debtor’s interest rate is smoothed out relative to the fully refinanced type of mortgage loan.

The description is not as detailed as for the fully refinanced loans. First, the actual market development seems to have concentrated almost exclusively on the fully refinanced type of loans, and the gradually refinanced type of loans is not offered by all the mortgage credit institutions. Second, among those that do the actual implementation appears to be treated with some degree of confidentiality. Hence, we are limited to a general description of possible solutions. In consideration of limitations as to a reasonable length of this paper we only describe possible ways for the initial funding problem and do not discuss lengthy details of possible implementations of the calculations for future refunding.

Assume that it has been agreed to refinance the fraction $1/n$ of the outstanding debt every year. This is carried out by issuing bonds with maturities $1, 2, \ldots, n$ and is known as a "P 100/n loan´´. In the first place this looks like a prepayment of $1/n$ of the loan, because it is irrelevant for the funding pattern whether the refinancing actually takes place or whether the fraction $1/n$ is simply prepaid. However, care must be taken in designing the funding profile, because the refinancing is a recurrent event. Too much funding in the "long end" prevents the planned prepayment/refinancing from taking place in the future – there are too few bonds maturing. Too much funding in the "short end" forces the actual prepayment/refinancing to be higher than planned – there are too many bonds maturing. The problem then is to construct a funding pattern, where

- there is an adequate amount of bonds of the shortest maturity outstanding to match the prepayment/refinancing
- there is an adequate amount of bonds with maturities $2, 3, \ldots$ outstanding so that the planned future prepayment/refinancing can take place without incurring negative funding problems

This cannot be guaranteed under all circumstances. The refinancing conditions in the market may develop in such a way that the planned future refinancing cannot be carried out. This will occur whenever there are already too many bonds of the nearest maturity outstanding in a refinancing situation. However, there are ways to do this in a manner that achieves this goal under relatively "smooth" market movements and comes very close to also under more volatile market movements.

An easy procedure for implementing a gradual refinancing scheme would be to build, say, a P33 loan as three F-loans with an obvious initialization procedure. This is easily implemented and easily calculated, although there will be some negligeable deviations from the planned refinancing schedule with 1/3 at each payment date. Given the simplicity this would have been a very simple solution. However, this has to our knowledge not been implemented by any of the credit institutions.

The original repayment profile $\{Z_1^d, Z_2^d, \ldots, Z_m^d\}$ is an element of the unit simplex $\mathbf{\Delta}^m$ in $\mathbb{R}^m$:

$$\mathbf{\Delta}^m \equiv \{x \in \mathbb{R}^m \mid \sum_{j=1}^{m} x_j = 1,\ x_j \geq 0,\ \ j = 1, 2, \ldots, m\}$$

One way is to calculate *as if* the loan is being repaid in its entirety within a given horizon, tacitly understood to be equal to n. Technically, the repayment profile from the corresponding non-adjustable rate loan is mapped onto the subsimplex $\mathbf{\Delta}^n$. Having done so, the previous calculations can be invoked to find the funding profile for the initial period.

There are infinitely many possibilities to construct this mapping onto $\mathbf{\Delta}^n$. We briefly mention two such candidates. We also briefly describe the essence of the algorithm for partially refinanced loans in EPO (1999), although it does not fall into this characterization.

One candidate, cf. (44), maintains the P 100/n-loan as a P 100/n-loan, but prepays the entire outstanding debt extraordinarily at time n. This way of calculating the funding profile will match the first prepayment, and it will have a tendency to push the funding principals towards the longer end.

$$\begin{bmatrix} Z_1^d \\ Z_2^d \\ \vdots \\ \vdots \\ Z_{m-1}^d \\ Z_m^d \end{bmatrix} \rightarrow \begin{bmatrix} Z_1^d + (1/n)(\sum_{t=2}^m Z_t^d) \\ ((n-1)/n))\left(Z_2^d + (1/n)\sum_{t=3}^m Z_t^d\right) \\ \vdots \\ ((n-1)/n))^{n-2}\left(Z_{n-1}^d + (1/n)\sum_{t=n}^m Z_t^d\right) \\ ((n-1)/n)^{n-1}\left(\sum_{t=n}^m Z_t^d\right) \end{bmatrix} \tag{44}$$

Each of the original repayments Z_p^d, $p = 2, \dots, n$, is repaid in accordance with the schedule:

$$\left(\frac{1}{n}, \frac{1}{n}\frac{n-1}{n}, \frac{1}{n}\left(\frac{n-1}{n}\right)^2, \dots, \frac{1}{n}\left(\frac{n-1}{n}\right)^{p-2}, \left(\frac{n-1}{n}\right)^{p-1}\right) \tag{45}$$

whereas the repayments Z_p^d, $p = n+1, n+2, \dots, m$ is repaid in accordance with the schedule valid for Z_n^d. It is straightforward to verify that

$$\frac{1}{n}\sum_{j=0}^{p-1}\left(\frac{n-1}{n}\right)^j + \left(\frac{n-1}{n}\right)^p = 1 \tag{46}$$

Another candidate, cf. (47), prepays exactly $1/n$ of outstanding debt in the beginning of each planning period of length n. In comparison with (44) it will push the funding principals towards the shorter end:

$$\begin{bmatrix} Z_1^d \\ Z_2^d \\ \vdots \\ \vdots \\ Z_{m-1}^d \\ Z_m^d \end{bmatrix} \rightarrow \begin{bmatrix} Z_1^d + (1/n)(\sum_{t=2}^m Z_t^d) \\ ((n-1)/n)\left(Z_2^d + (1/n-1)\right)\sum_{t=3}^m Z_t^d\right) \\ \vdots \\ (2/n)\left(Z_{n-1}^d + (1/2)(\sum_{t=n}^m Z_t^d)\right) \\ (1/n)\sum_{t=n}^m Z_t^d) \end{bmatrix} \tag{47}$$

Each of the original repayments Z_p^d, $p = 2, \dots, n$, is repaid in accordance with the schedule:

$$\left(\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n}, \frac{n+1-p}{n}\right) \tag{48}$$

whereas the repayments Z_p^d, $p = n+1, n+2, \dots, m$ is repaid in accordance with the schedule valid for Z_n^d.

The class of fully refinanced mortgage loans also falls into this category. The mapping is simple for these loans:

$$\begin{bmatrix} Z_1^d \\ Z_2^d \\ \vdots \\ \vdots \\ Z_{m-1}^d \\ Z_m^d \end{bmatrix} \rightarrow \begin{bmatrix} Z_1^d \\ Z_2^d \\ \vdots \\ Z_{n-1}^d \\ \sum_{t=n}^m Z_t^d \end{bmatrix} \tag{49}$$

Finally, we

A third candidate is found in EPO (1999). The idea can be described by the following steps.

Consider the actual sequence of repayments Z_t^d, which may well depend on the interest rate on the loan y, and assume that we do not take up any new funding, but continue to repay $1/n$ of the outstanding debt every year. Then the procedure fits a polynomial $\Gamma(t)$ of degree $n-1$ to describe the profile of the outstanding debt to be refinanced over the first n years:

$$\Gamma(t) = a_0 + a_1 t + a_2 t^2 + \ldots + a_{n-1} t^{n-1} \tag{50}$$

given that no new funding is issued. This is along the line of reasoning described in (44).

The coefficients a_p are uniquely determined by solving the system of equations:

$$\begin{bmatrix} 1 & 0 & 0 & \ldots & 0 \\ 1 & 1 & 1 & \ldots & 1 \\ 1 & 2 & 4 & \ldots & 2^{n-1} \\ 1 & 3 & 9 & \ldots & 3^{n-1} \\ \ldots & \ldots & \ldots & \ddots & \ldots \\ 1 & n-1 & (n-1)^2 & \ldots & (n-1)^{n-1} \end{bmatrix} \cdot \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_{n-1} \end{bmatrix} = \begin{bmatrix} \frac{1}{n} \\ \frac{1-Z_1^d}{n} \\ \vdots \\ \frac{1-\sum_{t=1}^{n-1} Z_t^d}{n} \end{bmatrix} \tag{51}$$

Having found this polynomium the procedure uses the same polynomial to desribe the profile of funding instruments Z_t^b. However, in order to make room for interest payments the procedure perturbs this polynomial by allowing for a change in the two coefficients of lowest order, i.e. a_0 and a_1. This introduces two new parameters γ_0 and γ_1. The interest rate y is the third variable to be determined.

The *funding requirement* is described by one equation with the two unknowns γ_0 and γ_1:

$$\sum_{j=1}^{n} k_j Z_j^b = 1 \qquad \Leftrightarrow \tag{52}$$

$$\sum_{j=1}^{n} k_j \left(\gamma_0 a_0 + \gamma_1 (j-1) + \sum_{p=2}^{n-1} a_p (j-1)^b \right) = 1 \qquad \Leftrightarrow \tag{53}$$

$$\gamma_0 a_0 \left(\sum_{j=1}^{n} k_j \right) + \gamma_1 a_1 \left(\sum_{j=2}^{n} k_j (j-1) \right) = 1 - \sum_{p=2}^{n-1} a_p \sum_{j=2}^{n} k_j (j-1)^b \tag{54}$$

The strict balance principle for the first year gives another equation with the two unknowns γ_0 and γ_1:

$$\sum_{j=1}^{n} c_j \left(\gamma_0 a_0 + \gamma_1 a_1 (j-1) + \sum_{p=2}^{n-1} a_p (j-1)^b \right) + a_0 = 1 \qquad \Leftrightarrow \tag{55}$$

$$\gamma_0 a_0 \left(1 + \sum_{j=1}^{n} c_j \right) + \gamma_0 a_1 \left(\sum_{j=2}^{n} c_j (j-1) \right) = 1 - \sum_{p=2}^{n-1} a_p \sum_{j=2}^{n} c_j (j-1)^b \tag{56}$$

The last equation is needed to determine the interest rate y as the discount rate that makes the sum of discounted payments equal to the unity. We rewrite this condition using Makeham's formula:

$$\sum_{j=1}^{n} P_j (1+y)^{-j} = 1 \qquad \Leftrightarrow \tag{57}$$

$$\sum_{j=1}^{n} \left(\gamma_0 a_0 + \gamma_1 a_1 (j-1) + \sum_{p=2}^{n-1} a_p (j-1)^b \right) \left[\frac{c_j}{y} + (1+y)^{-j} \left(1 - \frac{c_j}{y} \right) \right] = 1 \tag{58}$$

These three equations are simultaneous equations that can be solved by a variety of numerical methods. By construction the method does not produce negative funding. In EPO (1999) the procedure for recurrent refinancing follows the method described here, but it is important at every refunding to take care of the existing "old funding".

9 Appendix

<u>Proof of theorem 2, items (4) and (5)</u>

4. Since $a > 0$ the sign condition in (2) of theorem 2 is satisfied for $t = 1, 2, \ldots, n-1$. Although these payment profiles are portfolios of a bullet loan and a loan with payments $a^{t-1} P_1$ the portfolio weight depends on y. Hence the former portfolio argument is no longer valid.

   The present value relation fixes the choice of P_1 to be determined in such a way that

$$1 = P_1 \sum_{t=1}^{m} a^{t-1} (1+y)^{-t} = P_1 \frac{1}{a} \alpha_{\overline{m|}\left(\frac{1+y}{a}-1\right)} = P_1 \frac{1 - \left(\frac{1+y}{a}\right)^{-m}}{1+y-a} \quad \Rightarrow$$

$$P_1 = \frac{1+y-a}{1 - \left(\frac{1+y}{a}\right)^{-m}} \equiv \frac{1+y-a}{1-\phi^{-m}}, \quad \phi \equiv \frac{1+y}{a} \tag{59}$$

Hence the funding equation (31) becomes

$$1 = \left[\frac{1+y-a}{1-\phi^{-m}}\right]\sum_{t=1}^{n} a^{t-1}(m_t - m_{t-1}) + \left[\frac{1-\phi^{-(m-n)}}{1-\phi^{-m}}\right] a^n(m_n - m_{n-1}) \tag{60}$$

The payment P_1 is always increasing in y, independent of the positive parameter a. It is a times the annuity payment calculated with an interest rate $\phi - 1$. Hence the first term in (60) is increasing with y.

The second term in (60) is affected by the outstanding debt after n periods in a hypothetical annuity with m periods maturity. This outstanding debt is also increasing with y. Differentiating after y leads to

$$\frac{1}{a}\frac{(m-n)\phi^{-(m-n+1)}\left[1-\phi^{-m}\right] - \left[1-\phi^{-(m-n)}\right] m\phi^{-m-1}}{\left(1-\phi^{-m}\right)^2} \tag{61}$$

Obviously the sign is determined by the numerator alone. After cancellation of identical terms and the elimination of common positive factors the sign is determined by

$$\left(1-\frac{n}{m}\right)\left[1-\phi^{-m}\right] - \left[\phi^{-n}-\phi^{-m}\right] = \frac{n}{m}\phi^{-m} + \left[1-\frac{n}{m}\right]\phi^{-0} - \phi^{-n}$$

Because of the convexity of the function $x \to \phi^{-x}$ this expression is positive due to the subgradient inequality.

5. Since the coefficients in (31) are positive and decreasing, the condition guarantees monotonicity w.r.t. y. We only prove this for the case where a possible sign change in the sequence $\partial P_1/\partial y, \ldots, \partial P_n/\partial y$ occurs only once[19] at n'. The first partial derivatives up to n' are non-negative and the last ones are non-positive by assumption.

   The absence of negative forward rates is equivalent to the sequence of positive discount factors $m_1, m_2 - m_1, \ldots, m_n - m_{n-1}$ being decreasing. Hence we can evaluate the partial derivative:

$$\begin{aligned}
\frac{\partial \sum_{t=1}^{n} P_t[m_t - m_{t-1}]}{\partial y} &= \sum_{t=1}^{n} \frac{\partial P_t}{\partial y}[m_t - m_{t-1}] \\
&= \sum_{t=1}^{n'} \frac{\partial P_t}{\partial y}[m_t - m_{t-1}] + \sum_{t=n'+1}^{n} \frac{\partial P_t}{\partial y}[m_t - m_{t-1}] \\
&\geq [m_{n'+1} - m_{n'}] \sum_{t=1}^{n} \frac{\partial P_t}{\partial y} \;\; \geq 0
\end{aligned} \tag{62}$$

∎

Proof of theorem 3

[19] The general case can be proven in an entirely analogous way, but with considerable more notation.

$$\frac{1+c_n}{k_n} = (1+y)\left[1 - \sum_{j=1}^{n-1} \widetilde{Z}_j^d\right] \tag{63}$$

$$\frac{1}{k_n} + (1+y)\widetilde{Z}_{n-1}^d = \left[1 - \sum_{j=1}^{n-1} \widetilde{Z}_j^d\right] \tag{64}$$

Substitute from (63) into (64) to obtain

$$\widetilde{Z}_{n-1}^d = \frac{1+c_n}{k_n}(1+y)^{-2} - \frac{1}{k_n}(1+y)^{-1} \tag{65}$$

From (63) we can obtain an expression for $\widetilde{Z}_{n-1}^d$ in terms of y. Since the supplementary funding variables form a geometric series with factor $(1+y)$, (63) can be formulated as

$$\frac{1+c_n}{k_n} = (1+y)\left[1 - \widetilde{Z}_{n-1}^d \sum_{j=0}^{n-2}(1+y)^{-j}\right] \tag{66}$$

Upon substitution this results in:

$$1 - \frac{c_n}{k_n}\sum_{j=1}^{n-1}(1+y)^{-j} - \frac{1+c_n}{k_n}(1+y)^{-n} = 0$$

This polynomial has exactly one positive roots for $(1+y)$ according to Descartes' rule of signs.

For annuity loans with a balloon the argument is almost identical. The three equations to be examined are:

$$Z_n^b = \frac{1}{k_n} \tag{67}$$

$$\frac{1+c_n}{k_n} = (1+y)\left[1 + \sum_{j=1}^{n-1}(\widetilde{Z}_j^d - Z_j^d)\right] \tag{68}$$

$$\frac{1}{k_n} = (1+y)(\widetilde{Z}_{n-1}^d - Z_{n-1}^d) + \left[1 + \sum_{j=1}^{n-1}(\widetilde{Z}_j^d - Z_j^d)\right] \tag{69}$$

Since Z_j^d as well as $\widetilde{Z}_j^d$ follow geometric series with factor $1+y$ the argument is exactly the same as for the bullet loan.

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