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On Makeham's Formula and Fixed Income Mathematics

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Bjarne Astrup Jensen

INSTITUT FOR FINANSIERING, Handelshøjskolen i København
Solbjerg Plads 3, 2000 Frederiksberg C
tlf.: 38 15 36 15   fax: 38 15 36 00

DEPARTMENT OF FINANCE, Copenhagen Business School
Solbjerg Plads 3, DK - 2000 Frederiksberg C, Denmark
Phone (+45)38153615, Fax (+45)38153600
www.cbs.dk/departments/finance

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[1]Department of Finance, Copenhagen Business School, Rosenørns Alle 31, 3rd, DK-1970 Frb. C., Denmark. e-mail: fibj@cbs.dk. I thank Leif Hasager, Jari Käppi, Jørgen Aase Nielsen, Jesper Rangvid and Bjarne G. Sørensen for comments on an earlier version. I thank J.J. McCutcheon and Ms. Sally Grover for bibliographical assistance in tracking down the original publication by Makeham. The author gratefully acknowledges financial support from the Danish Social Science Research Council.

Abstract

The return on a bond investment comes from three sources: Interest payments, realized capital gains and accrued capital gains. We provide an exact description on how the capital gains can be measured under a variety of accounting rules for measuring accruals and study the theoretical properties of such rules, their taxation consequences and the relation between the yield before tax and the yield after tax. The vehicle of our exposition is Makeham's formula, an actuarial formula for the present value of a payment stream largely neglected in the finance literature.

Keywords

Makeham's formula, consistent accounting schemes, accrued capital gains, yield before tax and yield after tax.

1 Introduction

The predominant part of the literature on fixed income analysis does not pay much attention to the *variety* of accounting rules and taxation legislation found in different countries. Taxes are usually treated – if at all – by relying solely upon a US institutional setup and US legislation; in few cases the rules of other countries are used. And the accounting rules are generally considered as being outside the domain of financial theory.

This is an inappropriate view. In some countries, capital gains – and in particular unrealized capital gains – are an important source of reported income for institutional investors like pension funds and life insurance companies. The reported income is the foundation for the benefit accruing to the customers. Hence the exact way such accruals are being measured can have an impact on the distribution of bonus and also determine the value of the surrender option in life insurances, cf. e.g. Albizzati and Geman (1994) and Grosen and Jørgensen (1997). Additionally, the taxation of capital gains is often different from the taxation of interest income. Hence the exact way that realized and accrued capital gains enter the tax base is of interest.

In this paper we attempt to describe in general terms the working of a few different accounting and taxation rules. As the title suggests the vehicle for our exposition is *Makeham's formula*. Makeham's formula is a way of expressing the present value of a stream of certain payments in terms of the present value of its repayments instead of the usual expression of the present value of its payments. Makeham was an actuarian, mostly known for the mortality distribution carrying his name.[1] The formula was published in an actuarial context in 1875, cf. Makeham (1875). Although now 125 years old and although some attention is given to this formula in the actuarial literature,[2] the formula appears entirely ignored in the finance literature.

Makeham's formula is indeed useful in fixed income analysis. It provides a framework for studying the way that realized and accrued capital gains enter into the annual statement. Although an infinite variety of methods exists to account for accruals, one particular choice - the constant yield method - allows the yield of a bond to be interpreted as a return measure. Makeham's formula is also applicable in the analysis of taxation and the way that different taxation schemes affect the relation between the yield before tax, the actual yield after tax and its deviation from the equivalent yield after tax obtainable from investing in an otherwise identical par bond.

The disposition of the paper is as follows.

In section 2 we introduce the bond notation applied in this paper, where a general payment stream is considered. The relation between the outstanding principle and the payments follow a first order difference equation, and in section 3 the analogous relation and notation for non-par bonds is introduced.

In section 4 we derive Makeham's formula. Whereas the usual present value relation can be interpreted as a pricing relation utilizing the decomposition of the bond's payments into a sequence of zero-coupon bonds, Makeham's formula utilizes a decomposition of the bond's payments into bullet bonds. As will become clear below, this allows a clear description of the capital gains in terms of yield, coupon rate and the discounted sum of repayments.

[1] The Gompertz-Makeham mortality distribution, cf. e.g. Gerber (1997), p. 18.

[2] See e.g. the book McCutcheon and Scott (1986) and the paper Hossack and Taylor (1975), the latter published as a centenary celebration of the formula.

In section 5 we define what we call a consistent accounting scheme for the amortization of capital gains. Four examples that are known to be used in accounting and taxation legislation are given. One of these, the so-called "constant yield method", is given special attention in section 6.

Makeham's formula has various versions after tax, which are derived in sections 7-11 and used in the analysis of the relation between the yield before tax and the yield after tax. We treat different rules for the taxation of capital gains.

In section 7 the situation where capital gains are tax-free is treated. When capital gains are tax-free, high yields after tax – in the sense that they deviate from the yields obtainable from similar par bonds – are associated with both *large* capital gains and *quick* capital gains. The need for large capital gains is obvious. The need for quick capital gains is due to the annualization of the return measure - large capital gains become a negligible fraction of the annual return when the necessary holding period for these capital gains to be realized becomes large. Large capital gains are usually associated with long term bonds, and the capital gains of these bonds are not realized quickly.

In section 8 the case where capital gains are taxed upon repayment is treated. Here, two opposite forces are at work. Quickly realized capital gains through quick repayments lead to quickly realized tax payments with a high present value of the tax burden. Slowly realized capital gains through slow repayments lead to slowly realized tax payments with a low present value of the tax burden. However, slowly realized capital gains also lead to a low fraction of the annualized return coming from capital gains.

In section 9 taxation according to the constant yield method for measuring accruals is examined. Provided that "wash sales" and "timing options" can be eliminated this is exactly the taxation principle that leads to the relation between before tax yields and after tax yields known from par bonds.

In section 10 taxation according to the linear appreciation method for measuring accruals is examined. For reasonable parameter values this appreciation method produces yields after tax slightly below, but very close to that obtainable from investing in par bonds.

In section 11 valuation and taxation by mark-to-market valuation is discussed. Here it is not meaningful to measure a yield after tax, because the tax payments by their very nature are stochastic. Only the sum of the tax payments is known. However, the mark-to-market valuation principle is a neutral taxation system in the sense that no tax segmentation will occur. This is also the case for the "constant yield method", whenever rules can be enforced to rule out "wash sales" and "timing options".

Section 12 concludes the paper and suggests some generalizations of the analysis. We maintain throughout the assumption that the tax rate is the same for all sources of income that are part of the tax base. This is easily generalized. Although some of the magnitudes will change, the basic qualitative conclusions remain unchanged. We also demonstrate an application of Makeham's formula in a reversed version that produces an analytical formula for the yield in some cases.

2 Bond notation

In this paper the term "fixed income security" is understood as a bond with the following characteristics:

- Payments occur at equidistant points in time, indexed by $j = 1, 2, \ldots, n$. The payment at time j is denoted by P_j.
- The face value of the bond is denoted by OP_0 for "outstanding principal".
- The bond carries a nominal interest rate c, fixed throughout the life-time of the bond, referred to as the *coupon rate*.
- After each payment there is an outstanding principal, denoted by OP_j, that must be repaid during the remaining lifetime of the bond.
- Each payment consists of interest payments and repayments of the principal. The interest payments at time j are calculated from the previous period's outstanding principal OP_{j-1} as $c \cdot OP_{j-1}$. The remaining part of the payment P_j is the repayment of the principal in period j and is denoted by Z_j^p.

As a matter of definitions the following relations are true:

$$P_j = c \cdot OP_{j-1} + Z_j^p \tag{1}$$

$$OP_j = (1 + c) \cdot OP_{j-1} - P_j \tag{2}$$

$$OP_j = OP_{j-1} - Z_j^p \tag{3}$$

Immediately after the last payment at the maturity date n the outstanding principal is zero, i.e. $OP_n \equiv 0$. Hence the first-order difference equation (2) has as its uniquely determined solution the present value relation:

$$OP_j = \sum_{t=j+1}^{n} P_t \cdot (1 + c)^{-(t-j)} \tag{4}$$

with the usual present value relation for $j = 0$ as a special case.

Different bonds can be characterized either in terms of the shape of the sequence of payments $P_1, P_2, \ldots, P_n$ or in terms of the shape of the sequence of repayments $Z_1^p, Z_2^p, \ldots, Z_n^p$. Three types of payment patterns are frequently found in financial markets:[3]

1. Bullet bonds, where $Z_1^p = Z_2^p = \ldots = Z_{n-1}^p = 0$, $Z_n^p = OP_0$. Consequently, $P_j = c \cdot OP_0$, $j = 1, 2, \ldots, n-1$ and $P_n = (1 + c) \cdot OP_0$. Bullet bonds can be thought of as an annuity payment $c \cdot OP_0$ with an extra payment OP_0 added at maturity.
2. Annuity bonds, where $P_1 = P_2 = \ldots = P_n \equiv P$. For annuities it is well known that the payment is found by means of the annuity factor $\alpha_{\overline{n}|c}$ and that the repayments follow a geometric series:

$$P = \frac{c}{1 - (1 + c)^{-n}} \cdot OP_0 \equiv \alpha_{\overline{n}|c}^{-1} \cdot OP_0 \qquad Z_j^p = (1 + c) \cdot Z_{j-1}^p$$

3. Serial bonds, where $Z_1^p = Z_2^p = \ldots = Z_n^p = (1/n) \cdot OP_0$.

[3] These standard types of bonds belong to the class of "systematic loans" introduced by Hasager and Jensen (1990).

3 Non-par bonds

When a bond of the type described in section 2 is traded it will only by coincidence be valued at time 0 at the price OP_0 corresponding to the principal and at time j by OP_j corresponding to the outstanding principal. The market will price the bond by discounting each payment P_j by an appropriate discount factor $d_{j,t}$, which is the price at time j of a unit zero-coupon bond with maturity date t. The price reached by the market will, analogously to the outstanding principal OP_j, be denoted by V_j^m for *market value*. It is given by

$$V_j^m = \sum_{t=j+1}^{n} P_t \cdot d_{j,t} \tag{5}$$

The *yield* y_m is defined as the discount rate that equates the present value of the payments P_t with the market price V_0^m:

$$V_0^m = \sum_{t=1}^{n} P_t \cdot (1 + y_m)^{-t} \tag{6}$$

It is well known that for any payment profile $P_1 \geq 0, P_2 \geq 0, \ldots, P_n \geq 0$ with no sign changes the yield is a uniquely determined number in $(-1, \infty)$.

A bond is often quoted such that the market value V_0^m is expressed in terms of a percentage of the principal OP_0. For notational reasons it is easier to interpret this quotation as a decimal number. We will denote this number by k_0. The definitional relation is then

$$V_0^m \equiv k_0 \cdot OP_0 \tag{7}$$

However, once the bond has been purchased there are many paths that the book value during its lifetime can follow. We will return to this in section 5.

4 Makeham's formula

The usual present value formula (6) decomposes a given payment stream into a portfolio of zero-coupon bonds with principal values $P_j,\ j = 1, 2, \ldots, n$, and values the payment stream as the sum of the present value of each of these zero-coupon bonds. This is standard financial practice. Alternatively, Makeham's formula decomposes a given payment stream into a portfolio of bullet bonds according to the stream of repayments $Z_j^p,\ j = 1, 2, \ldots, n$, and values the payment stream as the sum of the present value of each of these bullet bonds.

Makeham's formula is the result of the following manipulation of the usual present value relation, using (1) and the fact that the outstanding principal is the sum of the remaining repayments. It is true for *any* choice of discount rate y, for which reason we denote the present value resulting from applying a given discount rate y as V_0^y. Among such discount rates is the yield y_m giving rise to the present value V_0^m.

$$
\begin{aligned}
V_0^y &= \sum_{t=1}^{n} P_t \cdot (1+y)^{-t} \\
&= \sum_{t=1}^{n} (c \cdot OP_{t-1} + Z_t^p) \cdot (1+y)^{-t} \\
&= \sum_{t=1}^{n} \left(c \cdot \sum_{q=t}^{n} Z_q^p + Z_t^p \right) \cdot (1+y)^{-t} \\
&= c \cdot \sum_{q=1}^{n} Z_q^p \cdot \left(\sum_{t=1}^{q} (1+y)^{-t} \right) + \sum_{t=1}^{n} Z_t^p \cdot (1+y)^{-t} \\
&= c \cdot \sum_{q=1}^{n} Z_q^p \cdot \frac{1-(1+y)^{-q}}{y} + \sum_{t=1}^{n} Z_t^p \cdot (1+y)^{-t} \\
&= \frac{c}{y} \cdot \left(\sum_{t=1}^{n} Z_t^p \right) + \left(1 - \frac{c}{y}\right) \cdot \sum_{t=1}^{n} Z_t^p \cdot (1+y)^{-t} \qquad (8) \\
&= \frac{c}{y} \cdot OP_0 + \left(1 - \frac{c}{y}\right) \cdot \sum_{t=1}^{n} Z_t^p \cdot (1+y)^{-t} \qquad (9)
\end{aligned}
$$

Expressions (8) and (9) are different ways of writing Makeham's formula.

Whenever $\sum_{t=1}^{n} Z_t^p = 1$ Makeham's formula in (8) gives right away an expression for k_0, which is the price per unit of principal when the discount rate is y:

$$
k_0 = \frac{c}{y} + \left(1 - \frac{c}{y}\right) \cdot \sum_{t=1}^{n} Z_t^p \cdot (1+y)^{-t} \qquad (10)
$$

A third variant of Makeham's formula is

$$
V_0^y = \sum_{t=1}^{n} Z_t^p \cdot \left(\frac{c}{y} + \left(1 - \frac{c}{y}\right) \cdot (1+y)^{-t} \right) \qquad (11)
$$

Equation (11) reflects the fact that the value of the bond is the value of a principal-weighted portfolio of bullet bonds. This is so because for the bullet bond, $Z_1^p = Z_2^p = \ldots = Z_{n-1}^p = 0$, $Z_n^p = OP_0$. Hence the pricing formula for the bullet bond becomes

$$
V_0^y = OP_0 \cdot \left[\frac{c}{y} + \left(1 - \frac{c}{y}\right) \cdot (1+y)^{-n} \right] \qquad (12)
$$

We conclude this section by showing the following relation (13), which is a corollary of Makeham's formula.

Theorem 1 *The capital gain $(1-k_0) \cdot OP_0$ can be written as*

$$
(1-k_0) \cdot OP_0 = (y-c) \cdot \sum_{j=1}^{n} Z_j^p \cdot \frac{1-(1+y)^{-j}}{y} = (y-c) \cdot \sum_{j=1}^{n} Z_j^p \cdot \alpha_{\overline{j|}y} \qquad (13)
$$

Hence it can be interpreted as the sum of periodic returns in excess of the coupon rate on the individual repayments in the sense that Z_j^p *earns this excess return in the first* j *periods, the present value of which is* $(y-c)\cdot Z_j^p \cdot \alpha_{\overline{j}|y}$. ▮

Proof *Assume wlog* $OP_0 = 1$. *From Makeham's formula it follows that*

$$\begin{aligned} 1-k_0 &= \left(1-\frac{c}{y}\right)\cdot\left[1-\sum_{j=1}^{n} Z_j^p\cdot(1+y)^{-j}\right] \\ &= \left(1-\frac{c}{y}\right)\cdot\left[\sum_{j=1}^{n} Z_j^p-\sum_{j=1}^{n} Z_j^p\cdot(1+y)^{-j}\right] \\ &= (y-c)\cdot\sum_{j=1}^{n} Z_j^p\cdot\frac{1-(1+y)^{-j}}{y} \\ &= (y-c)\cdot\sum_{j=1}^{n} Z_j^p\cdot\alpha_{\overline{j}|y} \end{aligned} \tag{14}$$

▮

5 Consistent accounting schemes

Buying a non-par bond entitles the owner to three different types of payments:

1. Interest payments (or "coupon payments")
2. Repayment of the purchasing price V_0^m
3. Capital gains in the total amount of $(1-k_0)\cdot OP_0 \equiv OP_0 - V_0^m$

A **consistent accounting scheme** is a way to account for the *return components*, i.e. items 1) and 3) above, including rules for measuring accruals. Interest payments are accounted for in accordance with the payment date in this paper, and we assume that these payment dates coincide in a natural way with accounting dates. In any case interest payments are treated in a fairly homogeneous way across countries.

This is not the case for accrued capital gains. Any scheme for measuring accruals involves a sequence of book values of the outstanding principals at any accounting date. This sequence of valuations can be expressed as $\{k_0, k_1, \ldots, k_n\}$, where

(i) k_0 is the purchasing price per unit of principal

(ii) $k_n = 1$

(iii) all k_t's are positive numbers

With this convention the outstanding principal at time t has book value $k_t \cdot OP_t$. The accounting return in any single period $(t-1,t]$ is denoted by AR_t and is given by

$$AR_t = \underbrace{c \cdot OP_{t-1}}_{\text{coupon payments}} + \underbrace{(1-k_{t-1}) \cdot Z_t^p}_{\text{repayment at par}} + \underbrace{(k_t - k_{t-1}) \cdot OP_t}_{\substack{\text{value adjustment of the} \\ \text{outstanding principal}}} \tag{15}$$

A set of book market values k_t together with (15) constitute our definition of a consistent accounting scheme.

Theorem 2 *Any consistent accounting scheme is a scheme for amortizing the total amount of capital gain $(1-k_0) \cdot OP_0$ through yearly accruals.* ▮

Proof *The following particular way of performing a double summation over the grid-points $(t,j) \in \{1,2,\ldots,n\} \times \{1,2,\ldots,n\}$ gives the result:*

$$\begin{aligned}
(1-k_0) \cdot OP_0 &= \sum_{t=1}^{n}(k_t - k_{t-1}) \cdot \sum_{j=1}^{n} Z_j^p \\
&= Z_1^p \cdot \sum_{t=1}^{n}(k_t - k_{t-1}) + (k_1 - k_0) \cdot \sum_{j=2}^{n} Z_j^p + \sum_{t=2}^{n}(k_t - k_{t-1}) \cdot \sum_{j=2}^{n} Z_j^p \\
&= Z_1^p \cdot (1-k_0) + (k_1 - k_0) \cdot OP_1 + \sum_{t=2}^{n}(k_t - k_{t-1}) \cdot \sum_{j=2}^{n} Z_j^p \qquad (16)
\end{aligned}$$

The same procedure can now be replicated successively for each of the terms in the remaining sum in (16) with k_0 replaced by k_1, k_2 and so forth and OP_0 replaced by OP_1, OP_2 and so forth. The resulting expression is

$$(1-k_0) \cdot OP_0 = \sum_{t=1}^{n} Z_t^p \cdot (1-k_{t-1}) + \sum_{j=1}^{n}(k_j - k_{j-1}) \cdot OP_j \tag{17}$$

▮

The capital gains associated with repayments at par are fully accounted for at the time of repayment. The coupon payments and the repayments at par are associated with cash flows, whereas the last term is an imputed amount.

Differences in accounting practice across countries, and also across different types of investors within a given country, are to a large extent captured by the following four valuation principles:

1. Accounting by market valuation or "mark-to-market valuation":
   k_t=actual market price at the end of the year, $k_n \equiv 1$

2. Accounting by repayment[4] or valuation by "historical acquisition cost":
   $k_{n-1} = k_{n-2} = \ldots = k_0, \; k_n \equiv 1.$

[4]Or accounting by realization.

3. Accounting by the "constant yield method":[5]

$$k_t \equiv \frac{V_t^{y_m}}{OP_t} \tag{18}$$

$$V_t^{y_m} \equiv \sum_{j=t+1}^{n} P_j \cdot (1+y_m)^{-(j-t)} \tag{19}$$

$$OP_t \equiv \sum_{j=t+1}^{n} P_j \cdot (1+c)^{-(j-t)} \tag{20}$$

where y_m is the yield measured at $t=0$.

4. Accounting by linear appreciation:

$$k_t = k_0 + \frac{t}{n} \cdot (1-k_0) \tag{21}$$

Accounting by mark-to-market valuation is usually considered as the true economic way of measuring returns in single periods. The argument against this in some situations is that a reported high volatility in returns over a sequence of years during a holding period does not necessarily mean a high volatility over the holding period as a whole. This may be relevant for individuals involved in pension savings plans, and in defined contribution pension saving schemes, based on actuarial principles, some smoothing mechanism is typically implemented.

Accounting by repayment is self-explanatory. The problem with this principle is that to the extent that the capital gain component is a significant part of the total return no investor is able to account for this return component in any "smooth" manner. Unless the maturity structure of the bonds in the portfolio is relatively even spread out there will be sizeable discontinuities in the reported returns that do not arise from market related variations in the term structure.

Valuation by the "constant yield method" is such a smoothing mechanism, but others could be considered. E.g. the linear amortization of the capital gain component analogous to the depreciation schemes found in some countries for long-lived investment assets.

Accounting by the "constant yield method" has some economically appealing properties. The next section describes the working of this principle. Before doing this we state the following general property for a consistent accounting scheme.

The assumption of a constant yield can be assumed for computational reasons without postulating anything about actual market behaviour. The sequence of values $V_j^{y_m}$ is also relevant for the amortization scheme for a *fixed rate loan* with principal $V_0^{y_m}$, funded by selling off the payments $P_1, P_2, \ldots, P_n$ as one pass-through bond. In a fixed rate loan with principal $V_0^{y_m}$ and payments $P_1, P_2, \ldots, P_n$, the debtor amortizes the loan in accordance with the following dynamics for the value of the outstanding principal, cf. (2):

$$V_j^{y_m} = V_{j-1}^{y_m} \cdot (1+y_m) - P_j \tag{22}$$

with the obvious terminal condition $V_n^{y_m} = 0$. The solution to this difference equation is precisely (19). And the payments P_t match exactly the payments to the bondholders in the form of coupon payments and repayments of the bond principal.

[5]In other contexts also termed "price change due to passage of time" or "maturity shortening".

6 Accounting by the "constant yield method"

Since the price $V_0^{y_m}$ is the solution to the difference equation shown in (22) the payments P_j can be interpreted in two ways:

- as an amortization of OP_0 by interest rate payments according to the interest rate c and repayments Z_j^p or
- as an amortization of $V_0^{y_m}$ by interest rate payments according to the interest rate y_m and (residually determined) repayments $Z_j^{y_m} \equiv V_{j-1}^{y_m} - V_j^{y_m}$ summing to $V_0^{y_m}$: $\sum_{j=1}^n Z_j^{y_m} = V_0^{y_m}$

As a consequence of this we have the following relation:

$$P_j = c \cdot OP_{j-1} + Z_j^p = y_m \cdot V_{j-1}^{y_m} + Z_j^{y_m} \tag{23}$$

Interpreting the yield as a return measure for the period $(j-1, j]$ can be obtained by a rewriting of (23), cf. also (15):

$$\begin{aligned}
AR_j &= c \cdot OP_{j-1} + (1 - k_{j-1}) \cdot Z_j^p + (k_j - k_{j-1}) \cdot OP_j \\
&= c \cdot OP_{j-1} + (OP_{j-1} - OP_j) - (k_{j-1} \cdot OP_{j-1} - k_j \cdot OP_j) \\
&= c \cdot OP_{j-1} + (OP_{j-1} - OP_j) - (V_{j-1}^{y_m} - V_j^{y_m}) \\
&= \underbrace{c \cdot OP_{j-1}}_{\text{coupon payments}} + \underbrace{Z_j^p - Z_j^{y_m}}_{\text{capital gains}} \\
&= \underbrace{y_m \cdot V_{j-1}^{y_m}}_{\text{reported return}}
\end{aligned} \tag{24}$$

In the second line of this derivation the valuation relations k_j deriving from the constant yield method, cf. (18)-(20), has been used. The yield y_m is thus the return earned in any period $(j-1, j]$ on the outstanding value $V_{j-1}^{y_m}$ when the accounting scheme derives from the constant yield principle.

The yield is a standard financial index number published daily in exchange listings from the bond market. It usually attracts much interest due to an implicit interpretation as a return measure on the initial investment over the entire horizon spanned by the maturity of the bond. The problems with this interpretation are treated in any standard textbook. The fact that it is possible to interpret the yield as a return measure, given that the accrued capital gains are appropriately accounted for in the sense defined above, is less well known.

In the general case look at the following variant of Makeham's formula:

$$y_m \cdot V_0^{y_m} = c \cdot OP_0 + (y_m - c) \cdot \sum_{j=1}^{n} Z_j^p \cdot (1 + y_m)^{-j} \tag{25}$$

The last term gives an explicit expression for the capital gains component $Z_1^p - Z_1^{y_m}$, revealing that the relative weight of capital gains is determined *jointly* by the difference between the yield and the coupon rate $(y_m - c)$ *and* the discounted value of the repayments. Hence, the difference between the coupon rate and the yield has in itself little informational content.

As one extreme the *consol bond* has the present value c/y_m, meaning that the discounted value of the repayments are zero - as a matter of fact there are no repayments at all. In this case the difference between the coupon rate and the yield can be anything - there will never be any accrued capital gain to report.

Examples

1. For a unit bullet bond $Z_1^p = Z_2^p = \ldots = Z_{n-1}^p = 0$ and $Z_n^p = 1$. Hence

$$\begin{aligned} y_m \cdot V_{t-1}^{y_m} &= c - Z_t^{y_m} \quad \Rightarrow & (26)\\ y_m \cdot \left[\frac{c}{y_m} + \left(1 - \frac{c}{y_m}\right) \cdot (1+y_m)^{-(n-t+1)}\right] &= c - Z_t^{y_m} \quad \Rightarrow & (27)\\ Z_t^{y_m} &= -(y_m - c) \cdot (1+y_m)^{-(n-t+1)} & (28)\end{aligned}$$

   For a bullet bond sold below par the repayments $Z_t^{y_m}$ recorded on the loan are negative. This reflects the fact that by the constant yield method the (negative) repayments $Z_t^{y_m}$ are mirror images of the price change due to passage of time. Since the two parties agree at the end that the repayment is OP_0, the difference $OP_0 - V_0^{y_m}$ must be added to the original proceeds of the loan paid to the debtor at time 0 during the life-time of the bond.

   The unrealized capital gains $Z_j^p - Z_j^{y_m} = -Z_j^{y_m}$ form a growing geometric series with growth factor $1+y_m$. This means that per unit principal of the bullet bond, the capital gain accrued in any given period declines rapidly with the time to maturity.

   Consider e.g. a bullet bond with $c = 5\%$, $k_0 = 0.75$ and $n = 40$. For this bond the yield is $y_m = 6.84\%$. Upon investing 75 in this bond the reported returns in periods 1, 6 and 39, respectively, are

$$\begin{aligned} y_m \cdot k_0 &= 5 + (k_1 - k_0) = 5 + 0.1371 & (29)\\ y_m \cdot k_5 &= 5 + (k_6 - k_5) = 5 + 0.1816 & (30)\\ y_m \cdot k_{38} &= 5 + (k_{39} - k_{38}) = 5 + 1.612 & (31)\\ & & (32)\end{aligned}$$

   In the initial periods the interest payment is around 30 times the accrued capital gain, whereas towards maturity the accrued capital gain accounts for roughly one fourth of the reported return.

   In case the bond was sold above par the situation is different. Then $y_m < c$, $OP_0 < V_0^{y_m}$ and $Z_t^{y_m} > 0$ in order to bring $V_0^{y_m}$ down to the face value OP_0 of the issued bonds at maturity.

2. A zero-coupon bond or pure discount bond is a special case of a bullet bond. The accrued capital gain is

$$Z_t^p - Z_t^{y_m} = y \cdot (1+y_m)^{-(n-t+1)}, \quad t = 1, 2, \ldots, n \qquad (33)$$

3. For the annuity bond it is true that the annuity principle applies to the bond with principal OP_0 and repayments Z_t^p as well as to the loan with principal $V_0^{y_m}$ and repayments $Z_t^{y_m}$. Hence the repayments both follow geometric series. The entire picture of the amortization is given in (36)-(38):

$$Z_t^p = P \cdot (1+c)^{-(n-t+1)} \tag{34}$$

$$Z_t^{y_m} = P \cdot (1+y_m)^{-(n-t+1)} \tag{35}$$

$$y_m \cdot V_{t-1}^{y_m} = c \cdot OP_{t-1} + (Z_t^p - Z_t^{y_m}) \quad \Rightarrow \tag{36}$$

$$y_m \cdot \alpha_{\overline{n-t+1|}y_m} \cdot P = c \cdot \alpha_{\overline{n-t+1|}c} \cdot P + (Z_t^p - Z_t^{y_m}) \quad \Rightarrow \tag{37}$$

$$Z_t^p - Z_t^{y_m} = P \cdot \left[(1+c)^{-(n-t+1)} - (1+y_m)^{-(n-t+1)}\right] \tag{38}$$

The difference between two geometrically growing series as shown in figure 1 may exhibit various patterns. The fastest growing series - for a discount annuity bond $P \cdot (1+y_m)^{-(n-t+1)}$ - also has the lowest starting value. Typically the two series will *diverge* in the beginning and converge as t approaches n. I.e. the capital gain component will typically first increase and then decrease. This is shown in figure 1 for the parameters $n=40$, $c=5\%$ and $y_m=7.31\%$, corresponding to a discount of 25%: $V_0^{y_m}=75,000=k_0 \cdot OP_0=0.75 \cdot 100,000$.

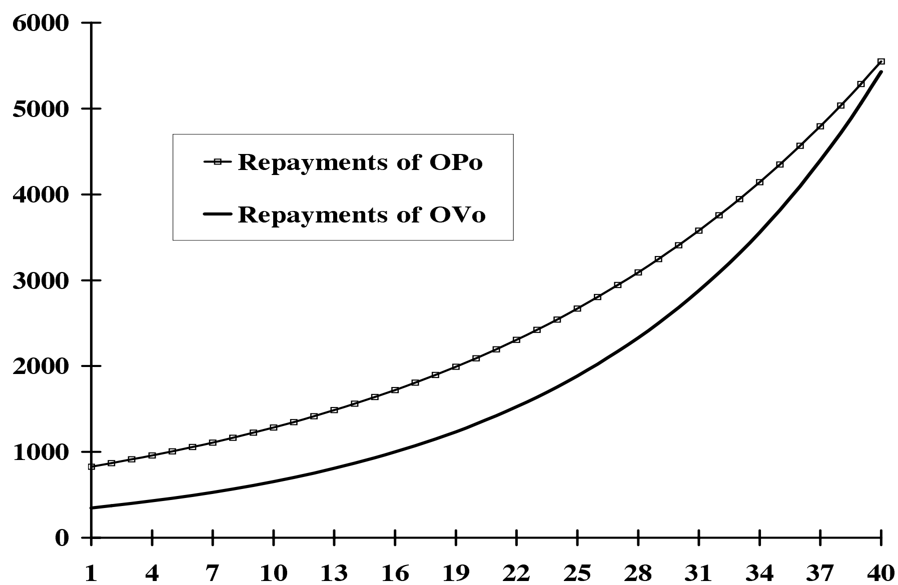

Figure 1: Repayments of OP_0 and V_0 for an annuity below par.
$OP_0=100.000$, $V_0=75.000$, $n=40$, $c=5\%$, $y_m=7.31\%$.

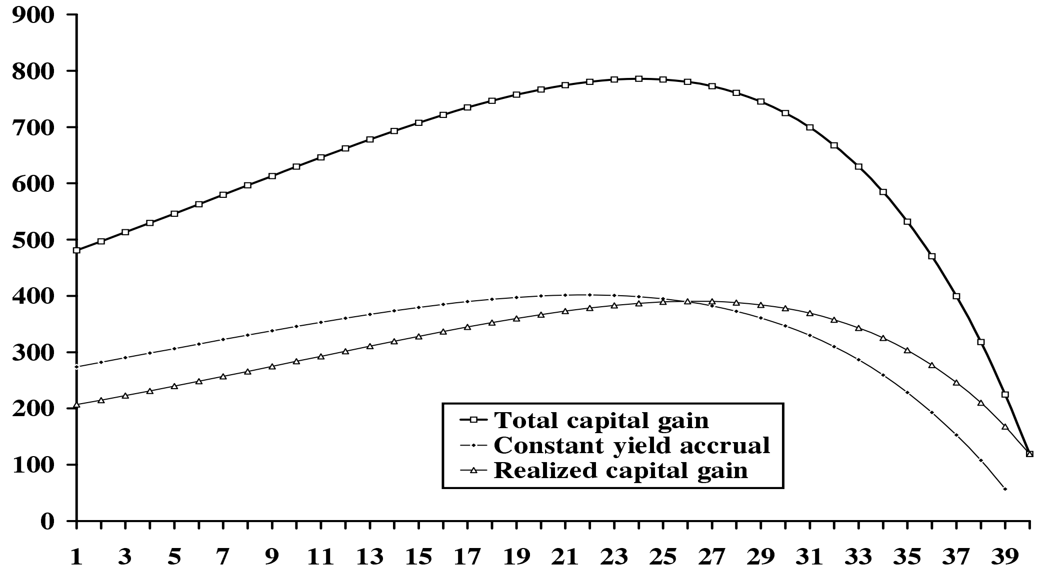

Figure 2: Amortization of capital gains for an annuity below par by the "constant yield method". $OP_0 = 100.000$, $V_0 = 75.000$, $n = 40$, $c = 5\%$, $y_m = 7.31\%$.

In figure 2 the capital gains are split into their realized and accrued parts. The humped shape of the curve for total capital gains are found in the curve for the realized part as well as in the curve for the accrued part.

7 Makeham's formula after tax: Tax free capital gains

To derive and interpret Makeham's formula after tax it is necessary to separate different rules for the taxation of realized and unrealized capital gains from each other. We assume throughout that interest payments are taxed linearly with a taxation rate T with tax payments falling due at the payment dates.

First consider the situation where all capital gains are tax free. And, symmetrically, capital losses not deductible. Makeham's formula is the result of the following manipulation, where the superscript $a.t.$ refers to "after tax" values:

$$
\begin{aligned}
k_0 &= \sum_{t=1}^{n} P_t^{a.t.} \cdot (1 + y^{a.t.})^{-t} \\
&= \sum_{t=1}^{n} (c \cdot (1-T) \cdot OP_{t-1} + Z_t^p) \cdot (1 + y^{a.t.})^{-t} \\
&= \sum_{t=1}^{n} \left(c \cdot (1-T) \cdot \sum_{q=t}^{n} Z_q^p + Z_t^p \right) \cdot (1 + y^{a.t.})^{-t}
\end{aligned}
$$

$$= \sum_{t=1}^{n} Z_t^p \cdot (1+y^{a.t.})^{-t} + c \cdot (1-t) \cdot \sum_{q=1}^{n} \sum_{t=1}^{q} Z_q^p \cdot (1+y^{a.t.})^{-t}$$

$$= \sum_{t=1}^{n} Z_t^p \cdot (1+y^{a.t.})^{-t} + c \cdot (1-T) \cdot \sum_{q=1}^{n} Z_q^p \cdot \frac{1-(1+y^{a.t.})^{-q}}{y^{a.t.}}$$

$$= \frac{c \cdot (1-T)}{y^{a.t.}} + \left(1 - \frac{c \cdot (1-T)}{y^{a.t.}}\right) \cdot \sum_{t=1}^{n} Z_t^p \cdot (1+y^{a.t.})^{-t} \tag{39}$$

In this case it is obvious that "the more capital gains, the better". Large capital gains can be obtained best by buying very long term bonds with low coupons and late repayments. But, as seen above, when the repayments of the principal take place very late the proportion of capital gains in the *annualized* return is small due to the discounted sum of repayments in Makeham's formula.

The difference between $y^{a.t}$ and $y_m \cdot (1-T)$ is governed by two factors. The one is the difference between y_m and c. The other is the discounted sum of the repayments Z_t^p. Assume that the bond is sold at a discount, i.e. $y_m > c$. Since

$$k_0 = \frac{c}{y_m} + \left(1 - \frac{c}{y_m}\right) \cdot \sum_{t=1}^{n} Z_t^p \cdot (1+y_m)^{-t} \tag{40}$$

the following inequality is valid:

$$\sum_{t=1}^{n} Z_t^p \cdot (1+y_m \cdot (1-T))^{-t} > \sum_{t=1}^{n} Z_t^p \cdot (1+y_m)^{-t} \quad \Rightarrow$$

$$k_0 < \frac{c \cdot (1-T)}{y_m \cdot (1-T)} + \left(1 - \frac{c \cdot (1-T)}{y_m \cdot (1-T)}\right) \cdot \sum_{t=1}^{n} Z_t^p \cdot (1+y_m \cdot (1-T))^{-t} \tag{41}$$

Since the rhs is a decreasing function of the discount rate y_m we have - not surprisingly - that $y^{a.t} > y_m \cdot (1-T)$. However, the closer k_0 is to c/y_m the less effect from the discounted sum of repayments and the closer $y^{a.t.}$ will be to $y_m \cdot (1-T)$.

From (40) it also follows that

$$y_m = \frac{c}{k_0} + \left(\frac{y_m - c}{k_0}\right) \cdot \sum_{t=1}^{n} Z_t^p \cdot (1+y_m)^{-t} \tag{42}$$

Having $k_0 < 1$ is equivalent to $y_m > c$. From (42) this implies that

$$y_m > \frac{c}{k_0} \quad \Rightarrow \quad y^{a.t} > y_m \cdot (1-T) > \frac{c \cdot (1-T)}{k_0}$$

We illustrate this by comparing three different types of bonds with the same coupon rate of 9% and the same yield to maturity before tax of 11%. The tax rate is fixed at 50%. Hence, for a par bond the yield after tax would be 5.5%.

For one period loans the name of the payment stream does not make any difference. They are identical. The resulting yield after tax is 6.417%, which is the maximum obtainable. For one period the lower bound is $c \cdot (1-T)/k_0$=4.5/0.9820=4.5825. At the other extreme - the consol bond - the yield after tax is 5.5% (=$(1-T) \cdot 11\%$), which is equal to the lower bound: 4.5/0.8182 = 5.5 and equal to the yield after tax obtainable in any other investment with a yield of 11% and no capital gains. In between the yield after tax is a monotonically decreasing function with time to maturity.

	Bullet loan		Annuity loan		Serial loan	
n	$y^{a.t.}$ in %	k_0	$y^{a.t.}$ in %	k_0	$y^{a.t.}$ in %	k_0
1	6,417	0.9820	6,417	0.9820	6,417	0.9820
2	6,378	0.9657	6,391	0.9735	6,392	0.9739
5	6,268	0.9261	6,313	0.9502	6,318	0.9526
10	6,109	0.8822	6,194	0.9177	6,211	0.9253
20	5,874	0.8407	5,992	0.8724	6,043	0.8906
30	5,726	0.8261	5,838	0.8462	5,923	0.8709
40	5,636	0.8210	5,726	0.8321	5,837	0.8589
50	5,581	0.8192	5,647	0.8248	5,775	0.8511
∞	5,500	0.8182	5,500	0.8182	5,500	0.8182

Table 1: The yield after tax as a function of the maturity for tax-free capital gains. $c = 9\%$, $y_m = 11\%$ and $T = 50\%$.

For any given maturity the yield after tax is ordered so that the serial loan has the highest obtainable yield after tax, followed by the annuity. The lowest obtainable yield after tax is found for the bullet bond.

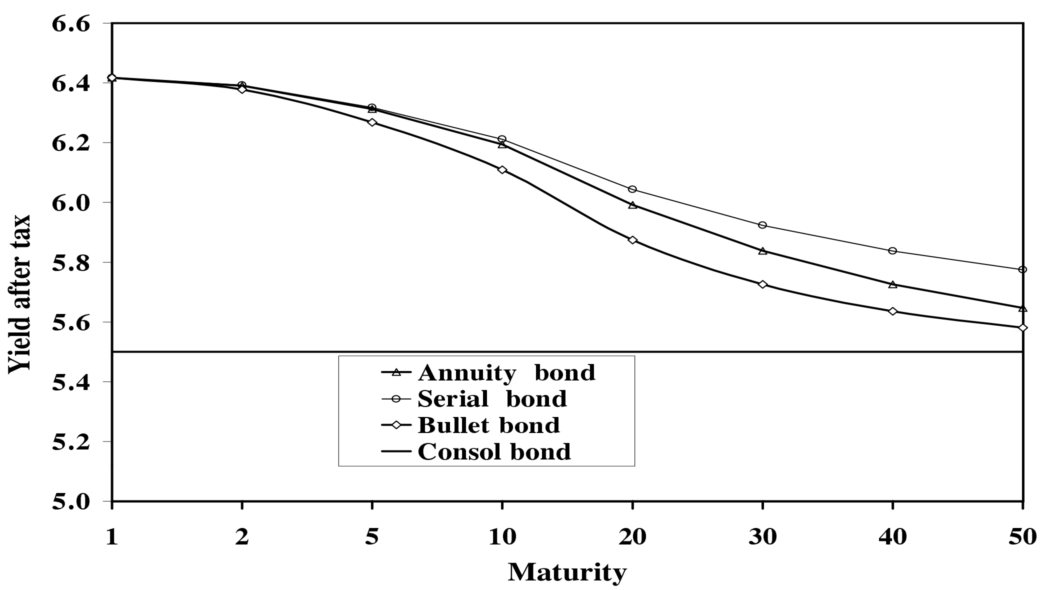

Figure 3: The yield after tax for tax free capital gains. c=9%, y_m=11% and T=50%.

8 Makeham's formula after tax: Capital gains taxed upon repayment

When capital gains are taxed upon repayment the payments $P_j^{a.t.}$ after tax are determined by (i) the tax on coupon payments and (ii) the repayments Z_j^p. The coupon payments are treated as usual, whereas the taxable capital gain is $(1-k_0)\cdot Z_j^p$. Hence the payments after tax become

$$P_j^{a.t.} = c \cdot (1-T) \cdot \sum_{t=j}^{n} Z_t^p + Z_j^p \cdot (1-(1-k_0)\cdot T) \tag{43}$$

Analogous to the derivation in (39) we obtain the following version of Makeham's formula:

$$k_0 = \frac{c\cdot(1-T)}{y^{a.t.}} + \left(1-(1-k_0)\cdot T - \frac{c\cdot(1-T)}{y^{a.t.}}\right)\cdot \sum_{j=1}^{n} Z_j^p \cdot (1+y^{a.t.})^{-j} \tag{44}$$

For all types of payment schedules the result is given in advance for $n=1$ and for $n=\infty$. The yield after tax follows the standard relation $y^{a.t} = y_m \cdot (1-T)$:

- For a one-period investment because taxation upon repayment is indistinguishable from a full and equal taxation of all financial returns, independent of the legislative classification applied to it.
- For the infinitely long lived security because in the limit all the securities become consol bonds. Hence repayments of the principal never take place.

The maximal obtainable yield after tax is to be found in between these two extremes for a maturity that depends on the type of bond in question and the relation between the coupon rate c and the yield y_m.

Again, two opposite forces are at work. On the one hand it is advantageous that the realized capital gains appear late in order for the tax payments to fall due late. On the other hand late realized capital gains mean that the content of capital gains in annualized terms become small.

It is possible to isolate k_0 in formula (44):

$$k_0 = \frac{1-T}{1-T\cdot\sum_{j=1}^{n} Z_j\cdot(1+y^{a.t.})^{-j}} \cdot \left[\frac{c}{y^{a.t.}} + \left(1-\frac{c}{y^{a.t.}}\right)\cdot\sum_{j=1}^{n} Z_j\cdot(1+y^{a.t.})^{-j}\right] \tag{45}$$

but this does not contribute to more precise estimates or interpretations concerning the magnitude $y^{a.t.}$.

We illustrate this taxation principle by using the same parameter values as used above for tax-free capital gains.

Note the scale of the axis in figure 4. The effect is indeed very small with maximum effect of 12 bp after tax for the bullet bond. The two forces - relatively large capital gains and the late taxation of capital gains - are truly counteracting.

The curve is hump-shaped under all circumstances, but the size and the location of the hump depends on the relation between y_m and c. The size of the hump increases and the decay towards $y_m \cdot (1-T)$ will be slower with an increasing distance between y_m and c. In the limit, when the bond becomes a zero-coupon bond, the curve will be monotonously increasing towards y_m, i.e. the yield after tax will converge towards the yield before tax when the maturity increases.

	Bullet loan		Annuity loan		Serial loan	
n	$y^{a.t.}$ in %	k_0	$y^{a.t.}$ in %	k_0	$y^{a.t.}$ in %	k_0
1	5,500	0.9820	5,500	0.9820	5,500	0.9820
2	5,523	0.9657	5,508	0.9735	5,508	0.9739
5	5,574	0.9261	5,530	0.9502	5,526	0.9526
10	5,617	0.8822	5,557	0.9177	5,554	0.9253
20	5,621	0.8407	5,583	0.8724	5,554	0.8906
30	5,591	0.8261	5,584	0.8462	5,550	0.8709
40	5,560	0.8210	5,571	0.8321	5,542	0.8589
50	5,538	0.8192	5,554	0.8248	5,534	0.8511
∞	5,500	0.8182	5,500	0.8182	5,500	0.8182

Table 2: The yield after tax as a function of the maturity for capital gains taxed upon repayment. $c = 9\%$, $y_m = 11\%$ and $T = 50\%$.

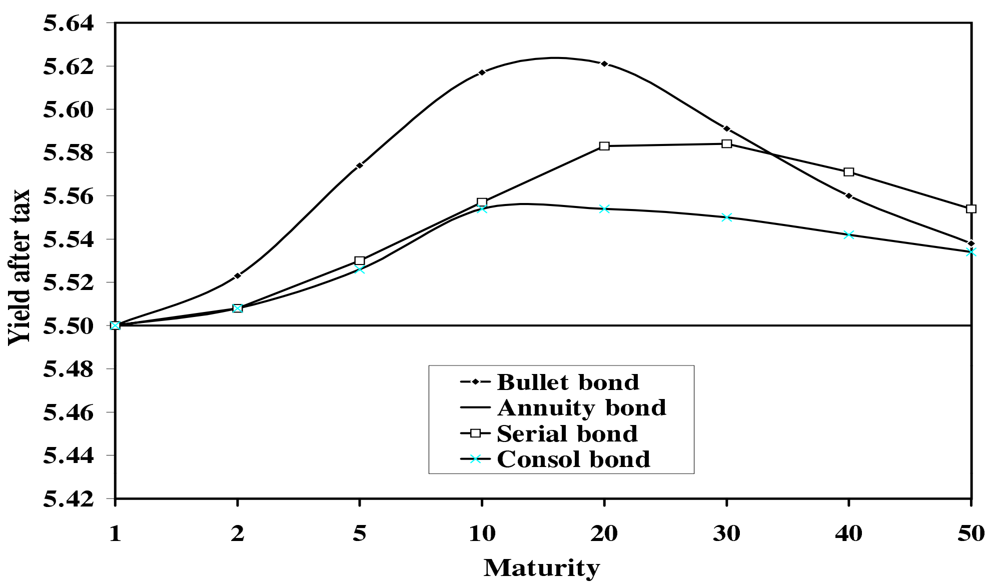

Figure 4: The yield after tax for capital gains taxed upon repayment. $c = 9\%$, $y_m = 11\%$.

9 Makeham's formula after tax: Capital gains taxed by the constant yield method

When capital gains are taxed by the constant yield method, the tax base in any period is given by the accounting return in (24). Hence it follows immediately that the accounting return after tax in any given period $(j-1, j]$ is given by $y_m \cdot (1-T) \cdot V_{j-1}^{y_m}$, i.e. the yield after tax is $y^{a.t.} = y_m \cdot (1-T)$. No further calculation is necessary.

This way of correcting for taxes in the discount factor is the standard textbook recipe. However, it only works when there are no capital gains or when capital gains are taxed according to the constant yield principle. It should be added that this principle requires that tax laws are eliminating the *timing option*[6] that arises from the ability to sell bonds before their maturity has been reached. If the tax laws allow, or even require, investors to enter the difference between the sales value and the book value into the tax base immediately upon realization, investors can intentionally generate an immediate tax rebate whenever market prices have dropped below book values. When the assets sold are being repurchased simultaneously this round-trip transaction is known as a *wash sale*.

In general the tax consequences are - as the point of departure - settled immediately and finally upon realization of a bond. Wash sale opportunities are to some extent limited in the tax laws in some countries.[7] The limitations may be in force to prevent round-trip transactions or to prevent the situation where "essentially similar assets are repurchased". A theoretical implementation that eliminates such opportunities would be to treat any sale of a bond before maturity as a *new* short position. This means that the buyer of a bond will carry the tax burden of the capital gains throughout the entire lifetime of the bond, but offsetting entries in the taxable income occurs if it is sold before its maturity date. The offsetting entries are calculated according to the yield valid at the date of sale without any reference to the yield valid at the date of acquisition. However, it is difficult to imagine a practical implementation of such principles that is able to eliminate tax evasion opportunities due to the use of other financial instruments to create synthetic bonds.

However, assuming that timing options and wash sale opportunities are somehow eliminated the taxation by the constant yield method has the property that *valuation* is independent of the tax rate. When $\{V_0^{y_m}, V_1^{y_m}, V_2^{y_m}, \ldots, V_{n-1}^{y_m}, 0\}$ is the solution to the difference equation (22) before tax it will also be the solution to the same difference equation with the payments P_j substituted by after tax payments $P_j^{a.t.}$ and the yield y_m before tax substituted by $y_m \cdot (1-T)$ - independent of the tax rate T. Hence, the taxation principle is *neutral*. One practical implication of this is that investors can value assets as if they were tax free - the effect of taxation on payments and on the discount factors cancels out.

[6]See e.g. Constantinides (1983).

[7]Chapter 9 in Fabozzi (1996) provides a short description of the central rules in US tax laws.

10 Makeham's formula after tax: Capital gains taxed by linear appreciation

When capital gains are taxed by linear appreciation the relevant entries in (15) are given by:

$$k_j = k_0 + \frac{j}{n} \cdot (1 - k_0) \tag{46}$$

$$k_j - k_{j-1} = \frac{1}{n} \cdot (1 - k_0) \tag{47}$$

$$1 - k_j = (1 - k_0) \cdot \left(1 - \frac{j}{n}\right) \tag{48}$$

The payments after tax become

$$\begin{aligned} P_j^{a.t.} &= c \cdot (1-T) \cdot OP_{j-1} + Z_j \cdot [1 - T \cdot (1 - k_{j-1})] - T \cdot (k_j - k_{j-1}) \cdot OP_j \\ &= [c \cdot (1-T) - T \cdot (k_j - k_{j-1})] \cdot OP_{j-1} + Z_j \cdot [1 - T \cdot (1 - k_j)] \\ &= \left[c \cdot (1-T) - T \cdot \frac{1}{n} \cdot (1 - k_0)\right] \cdot OP_{j-1} + Z_j \cdot \left[1 - T \cdot \left(1 - \frac{j}{n}\right)(1 - k_0)\right] \end{aligned} \tag{49}$$

The details of the derivation of Makeham's formula for this taxation principle is entirely analogous to the previous ones. The final result is:

$$\begin{aligned} k_0 &= \sum_{t=1}^{n} P_t^{a.t.} \cdot (1 + y^{a.t.})^{-t} \\ &= \frac{h}{y^{a.t.}} + \left(1 - T \cdot (1 - k_0) - \frac{h}{y^{a.t.}}\right) \cdot \sum_{t=1}^{n} Z_t^p \cdot (1 + y^{a.t.})^{-t} + \\ &\quad T \cdot (1 - k_0) \cdot \sum_{t=1}^{n} Z_t^p \cdot \frac{t}{n} \cdot (1 + y^{a.t.})^{-t} \end{aligned} \tag{50}$$

where

$$h = c \cdot (1 - T) - \frac{T}{n} \cdot (1 - k_0) \tag{51}$$

We show the results in table 3 for the same parameter values as used above.

The yields after tax are all less than, but remarkably close to the yield after tax obtainable from an otherwise identical par bond. This is due to the fact that the increase in book value according to the constant yield method always follows a progressively increasing curve for a below par bond. This convex feature tends to tax the accrued capital gains too much in the beginning and too little towards maturity, rendering the present value of the tax burden "too high". This effect is close to being negligible for the given moderate choice of parameters. The downward bias increases with an increasing distance between y_m and c, and in extreme cases the yield after tax may become negative.

	Bullet loan		Annuity loan		Serial loan	
n	$y^{a.t.}$ in %	k_0	$y^{a.t.}$ in %	k_0	$y^{a.t.}$ in %	k_0
1	5.500	0.9820	5.500	0.9820	5.500	0.9820
2	5.499	0.9657	5.496	0.9735	5.496	0.9739
5	5.491	0.9261	5.491	0.9502	5.491	0.9526
10	5.472	0.8822	5.485	0.9177	5.484	0.9253
20	5.426	0.8407	5.466	0.8724	5.468	0.8906
30	5.392	0.8261	5.444	0.8462	5.453	0.8709
40	5.373	0.8210	5.423	0.8321	5.443	0.8589
50	5.366	0.8192	5.416	0.8248	5.437	0.8511
∞	5,500	0.8182	5,500	0.8182	5,500	0.8182

Table 3: The yield after tax as a function of the maturity for capital gains taxed by linear appreciation. $c = 9\%$, $y_m = 11\%$ and $T = 50\%$.

This effect is stronger the later the capital gains are being realized upon repayment, because the capital gains tax must be paid anyway on a capital gain realized by repayment. For any given maturity it is the case that - with almost vanishing differences - the serial loan has the lowest yield after tax and the bullet loan the highest.

11 Valuation and taxation by mark-to-market valuation

The neutrality property is shared by the principle of taxation according to mark-to-market values. When the mark-to-market valuation principle is used the accounting returns are no longer predictable:

$$AR_t = \underbrace{c \cdot OP_{t-1}}_{\text{coupon payments}} + \underbrace{(1 - k_{t-1}) \cdot Z_t^p}_{\text{repayment at par}} + \underbrace{(k_t - k_{t-1}) \cdot OP_t}_{\substack{\text{market value adjustment of} \\ \text{the outstanding principal}}} \tag{52}$$

Here timing options and wash sales are eliminated by construction, but by the very nature of the valuation principle the concept of a yield after tax is difficult to define. The calculation of a yield after tax is based upon knowledge of the payment stream after tax, which by construction is stochastic in this case.

Let $V_1^m, V_2^m, \ldots, V_{n-1}^m, V_n^m \equiv 0$ be the sequence of market values which become equal to the book values under this accounting principle. In a before tax setting the following transactions have the same present value:

1. Buying the payment stream at time 0 at the price V_0^m.
2. Buying the payment stream at time 0 *and* selling the payment stream forward with delivery date at time 1 at the usual forward price which gives the forward contract value 0.

We adopt the notation that the initial one period rate of interest is y_0^1. Given this notation the last financial position is shown below, where the payment stream is split into two parts. The left part is the acquisition of the payment stream at the market price V_0^m, which gives the investor the first payment P_1 as well as the stochastic market value V_1^m of the remaining payments at time 1. The right part is the forward contract.

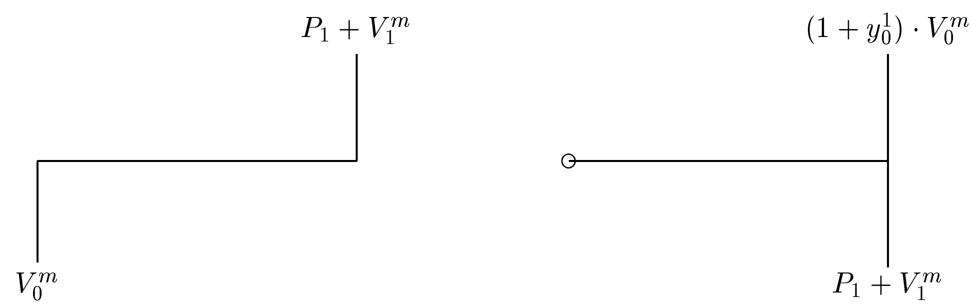

Figure 5

Adding these two positions together gives the following payment stream:

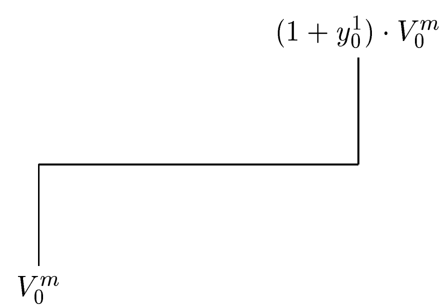

Figure 6

I.e. the investment has been transformed into a risk-free, one period investment earning the spot rate of interest y_0^1. Provided the forward contract is taxed in the same manner as the "underlying asset" the payment streams after tax are shown in figure 7 below.

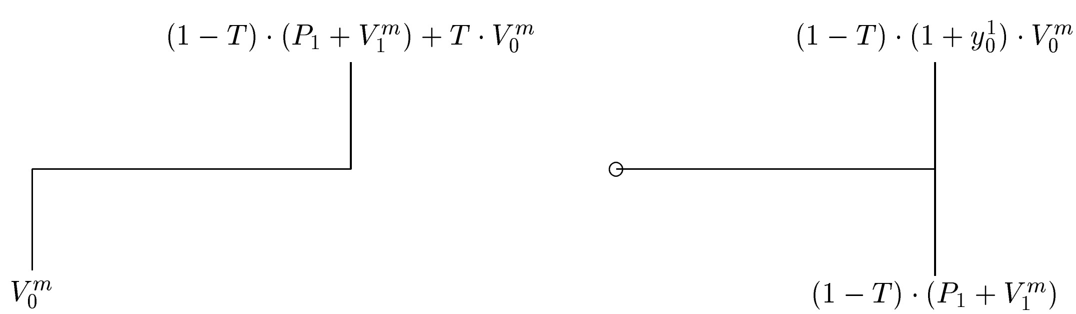

Figure 7

The resulting payment after tax is shown in 8. Since any other alternative with the same one year horizon will be taxed after this one year, the after tax discount factor is obviously $y_0^1 \cdot (1 - T)$. Hence the value V_0^m is agreed upon by every investor, independent of the tax rate T.

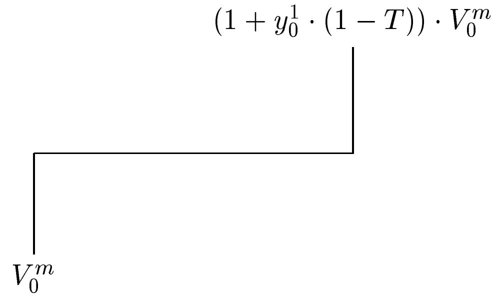

Figure 8

12 Makeham's formula reversed

Makeham's formula can be generalized in many ways, and other types of taxation rules than the ones given in this paper are possible. Introducing separate tax rates for interest payments and for capital gains is an immediate generalization. In Hossack and Taylor (1975) Makeham's formula is derived for the situation where the redemption values of the repayments are different from par value. Similar variants on this theme can be thought of.

We conclude this paper by showing an application of Makeham's formula to a class of loans, where a closed form solution for the yield can be found.

Consider a mortgage loan to be financed by issuing a pass-through bond, where the repayments are determined by the debtor's loan. I.e. the debtor repayment profile $Z_1^y, Z_2^y, \ldots, Z_n^y$ is given, whereas the repayment profile on the bond is endogenous. In order to finance one unit of such a loan, the face value of the bond must be $1/k_0$. If the coupon rate is c, a mirror image of Makeham's formula then gives:

$$\frac{1}{k_0} = \frac{y}{c} + \left(1 - \frac{y}{c}\right) \cdot \sum_{t=1}^{n} Z_j^y \cdot (1+c)^{-t} \tag{53}$$

Solving for y we arrive at

$$y = \frac{1 - \sum_{j=1}^{n} Z_j^y \cdot (1+y)^{-j}}{\frac{1}{k_0} - \sum_{j=1}^{n} Z_j^y \cdot (1+y)^{-j}} \tag{54}$$

If the debtor's repayment profile is independent of y, which is the case for a bullet loan as well as a serial loan[8], (54) gives an analytical solution for yield in terms of the bond price.

[8]But *not* for the annuity.

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