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DYNAMIC OPTIMIZATION IN DISCRETE TIME: LAGRANGE VERSUS THE HAMILTONIAN

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Dynamic Optimization in Discrete Time: Lagrange versus the Hamiltonian

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Abstract

This note illustrates the connection between the Langrangean function, applied to a standard dynamic optimization problem, and the Hamiltonian. I also derive the first order conditions under two different definitions of the Hamiltonian function.

1 The problem

Consider the following standard dynamic optimization problem. The management of a business enterprise sets out to maximize, by choosing investment and thereby the stock of capital, the market value of the firm's equity, V_{t-1}. That is,

$$\max_{\{I_s\}_{s=t}^{\infty}} V_{t-1} = \sum_{s=t}^{\infty} R_{t-1,s} div_s \tag{1}$$

subject to

$$K_s = I_s + (1-\delta)K_{s-1} \tag{2}$$

Accordingly, the manager selects a time path of investment, $\{I_s\}_{s=t}^{\infty}$, so as to maximize V_{t-1}. Market value, in turn, is given by equation (1) and equals the present value of

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future dividends, div_s, discounted back to time $t-1$ using the compound interest factor, $R_{t-1,s}$. Equation (2) shows how the capital stock of the firm evolves over time, the rate of physical depreciation being δ.

In the next section I set up the Lagrangean for this infinite horizon problem. I then show how this function relates to the Hamiltonian familiar from continuous time optimal control theory and derive the optimality conditions. In section 3 I derive the necessary conditions in the case where the Hamiltonian is defined in a slightly different way.

2 The Lagrangean and the flow-Hamiltonian

The Lagrangean for the problem (1)-(2) reads

$$\mathcal{L}_{t-1} = \sum_{s=t}^{\infty} \left\{ R_{t-1,s} div_s - R_{t-1,s} q_s \Big[K_s - I_s - (1-\delta) K_{s-1} \Big] \right\} \tag{3}$$

which I arrive at by combining (1) with the side constraint (2) for each period $s \geq t$. Hence, there is an infinity of side constraints to take into account, namely one representing the evolution of the stock of capital in each period. True to tradition in neo-classical investment theory, I let q_s denote the period s shadow price of capital. Now, rearranging the terms inside the square bracket yields

$$\mathcal{L}_{t-1} = \sum_{s=t}^{\infty} \left\{ R_{t-1,s} div_s - R_{t-1,s} q_s \Big[(K_s - K_{s-1}) - (I_s - \delta K_{s-1}) \Big] \right\} \tag{4}$$

and further

$$\mathcal{L}_{t-1} = \sum_{s=t}^{\infty} \left\{ R_{t-1,s} div_s + R_{t-1,s} q_s \left(I_s - \delta K_{s-1} \right) - R_{t-1,s} q_s \left(K_s - K_{s-1} \right) \right\} \tag{5}$$

Consider the first two terms inside the curly brackets in (5). The first one is period s dividends, discounted back to time $t-1$. The second term is the increase in the stock of productive capital times the shadow price of capital q_s, also discounted back to time $t-1$. Hence, these two terms indicate the contribution to the value of the firm deriving from its period s activities. Obviously, these activities comprise, on the one hand, production and investment which, through assumptions regarding financial structure and dividend policy, yields the flow of dividends, div_s, and, on the other hand, the change in the capital stock.

In a way similar to the continuous time case, I now define the Hamiltonian as precisely the sum of these two contributions. That is,

$$\mathcal{H}_s \equiv R_{t-1,s} div_s + R_{t-1,s} q_s \left(I_s - \delta K_{s-1} \right) \tag{6}$$

For later reference, I will denote $\mathcal{H}_s$ the "flow-Hamiltonian". Inserting this in (5) allows me to simplify to obtain

$$\mathcal{L}_{t-1} = \sum_{s=t}^{\infty} \left\{ \mathcal{H}_s - R_{t-1,s} q_s \left(K_s - K_{s-1} \right) \right\} \tag{7}$$

Next, I write out the first three terms under the summation sign, i.e. the terms relating to periods t, $t+1$ and $t+2$. This gives

$$\begin{aligned} \mathcal{L}_{t-1} = {} & \mathcal{H}_t - R_{t-1,t} q_t \left(K_t - K_{t-1} \right) \\ & + \mathcal{H}_{t+1} - R_{t-1,t+1} q_{t+1} \left(K_{t+1} - K_t \right) \\ & + \mathcal{H}_{t+2} - R_{t-1,t+2} q_{t+2} \left(K_{t+2} - K_{t+1} \right) \\ & + ... \end{aligned} \tag{8}$$

Notice now, that the capital stock in periods t,$t+1$,$t+2$,... appears *twice* in the Lagrangean. Combining these terms, and collecting the Hamiltonians under the summation sign, then yields

$$\mathcal{L}_{t-1} = R_{t-1,t} q_t K_{t-1} + \sum_{s=t}^{\infty} \mathcal{H}_s \tag{9}$$

$$+ \left(R_{t-1,t+1} q_{t+1} - R_{t-1,t} q_t \right) K_t \tag{10}$$

$$+ \left(R_{t-1,t+2} q_{t+2} - R_{t-1,t+1} q_{t+1} \right) K_{t+1}$$

$$+ ...$$

Consider the terms inside soft brackets. For some given K_s, $s \geq t$ this is

$$\Delta \left(R_{t-1,s+1} q_{s+1} \right) \equiv R_{t-1,s+1} q_{s+1} - R_{t-1,s} q_s$$

The economic interpretation of this term is that of a capital gain; it expresses the change in the shadow value of capital held through period $s+1$. Multiplying this by the amount of capital available at the end of period s yields the total capital gain on the stock of productive assets held by the firm during period $s+1$. Collecting these capital gains terms under a separate summation sign yields the simplified Lagrangean

$$\mathcal{L}_{t-1} = R_{t-1,t} q_t K_{t-1} + \sum_{s=t}^{\infty} \mathcal{H}_s + \sum_{s=t}^{\infty} \Big\{ \Delta \left(R_{t-1,s+1} q_{s+1} \right) K_s \Big\} \tag{11}$$

Now, the firm's management will seek to choose investment optimally from period t onwards. Therefore the first term on the right hand side in (11) does not affect optimal behavior, since it depends solely upon the stock of capital inherited from the past, K_{t-1}. I thus drop it from the Lagrangean and define

$$\mathcal{L}^*_{t-1} \equiv \sum_{s=t}^{\infty} \mathcal{H}_s + \sum_{s=t}^{\infty} \Big\{ \Delta \left(R_{t-1,s+1} q_{s+1} \right) K_s \Big\} \tag{12}$$

Before deriving the first order conditions characterizing optimal investment behavior, it may be worthwhile to consider the intuitive content of equation (12). According to $\mathcal{L}^*_{t-1}$ above, we may think of the firm as maximizing the sum of two components. The first one, the sum of the flow-Hamiltonians, gives (the present value of) the return in each period that results from decisions *within* those periods. As already explained, this comprises dividends, in turn reflecting productive activity and investment expenditures, and additions to the capital stock, i.e. net investment. The second component is the present value of the capital gains on the *existing* capital stock in each period.

It is now straightforward to derive the first order conditions characterizing the optimal choice of investment and capital. These are obtained by setting to zero the first derivatives of $\mathcal{L}^*_{t-1}$. Then

$$\frac{\partial \mathcal{L}^*_{t-1}}{\partial I_s} = 0 \Rightarrow \frac{\partial \mathcal{H}_s}{\partial I_s} = 0 \tag{13}$$

and

$$\frac{\partial \mathcal{L}^*_{t-1}}{\partial K_s} = 0 \Rightarrow \frac{\partial \mathcal{H}_{s+1}}{\partial K_s} = -\Delta \left(R_{t-1,s+1} q_{s+1} \right) \tag{14}$$

Equation (13) may be used to derive the familiar Tobin's q investment demand relation, while repeated forward substitution of (14) yields the shadow price q as the present value of the marginal product of capital. Equations (13)-(14) provide the discrete time analog of the necessary conditions well-known from, e.g., Dorfman (1969). They also show how these conditions may be conveniently expressed in terms of the flow-Hamiltonian, $\mathcal{H}$. That is, it is not necessary to set up the complicated Lagrangean in equation (3), one may economize on algebra and characterize optimal investment policy using the flow-Hamiltonian defined in (6) and the first order conditions in (13)-(14). In a certain sense it is quite intuitive why using the Hamiltonian approach facilitates computational ease. As pointed out above, this approach implies that one focuses directly on the actions that the firm in fact undertakes in each period. That is, production and investment. Of course, the change in the shadow value of the existing stock of capital is also part of the return to the firm's equity holders, but this is conveniently taken into account through the capital gains term on the left hand side of equation (14). Hence, while carrying out the same optimization, one avoids notational clutter by using the Hamiltonian. Equations (13)-(14) are similar to the first order conditions derived by Dixit (1990).[1] In the next section I derive necessary first order conditions using a slightly different definition of the Hamiltonian function.

3 The Lagrangean and the stock-Hamiltonian

In the previous section, I derived the first order optimality conditions, and expressed those in terms of the first derivatives of the Hamiltonian. This was accomplished through manipulating the Lagrangean. In this section, I take a slightly different route which leads naturally to defining what I will call the "stock-Hamiltonian". Again, the starting point is equation (3). The square bracket term may now be rearranged to give

$$\mathcal{L}_{t-1} = \sum_{s=t}^{\infty} \left\{ R_{t-1,s} div_s - R_{t-1,s} q_s \Big[K_s - (I_s + (1-\delta) K_{s-1}) \Big] \right\} \tag{15}$$

from which I obtain

$$\mathcal{L}_{t-1} = \sum_{s=t}^{\infty} \left\{ R_{t-1,s} div_s + R_{t-1,s} q_s \Big[I_s + (1-\delta) K_{s-1} \Big] - R_{t-1,s} q_s K_s \right\} \tag{16}$$

Next, I define the stock-Hamiltonian

$$\tilde{\mathcal{H}}_s \equiv R_{t-1,s} div_s + R_{t-1,s} q_s \Big[I_s + (1-\delta) K_{s-1} \Big] \tag{17}$$

[1]See Dixit (1990), equations 10.5 and 10.11.

Notice how $\tilde{\mathcal{H}}$ differs from the flow version defined in equation (6). The stock-Hamiltonian comprises dividends in period s, just like $\mathcal{H}$ does. However, the second term is now the capital stock at the end of the period multiplied by its shadow price. That is, by definition, equal to the value of the entire stock of productive capital possessed by the firm at the end of period s. In contrast, the flow version includes only *additional* capital accumulated during period s. Therefore, it is natural to refer to $\mathcal{H}_s$ as the flow-Hamiltonian, because it expresses the contribution to the value of the firm that derives from its period s operations. In contrast, the stock version of the Hamiltonian, $\tilde{\mathcal{H}}_s$, includes the dividend return in period s plus the value of the entire capital stock at the end of the period. Hence, $\tilde{\mathcal{H}}_s$ captures the contribution to market value arising from activities beginning in period s and extending into the (indefinite) future. Inserting the definition of $\tilde{\mathcal{H}}_s$ into (16) yields

$$\mathcal{L}_{t-1} = \sum_{s=t}^{\infty} \left\{ \tilde{\mathcal{H}}_s - R_{t-1,s} q_s K_s \right\} \tag{18}$$

As opposed to (12), this equation has no readily available interpretation. Differentiating with respect to I_s and K_s yields the necessary first order conditions

$$\frac{\partial \mathcal{L}_{t-1}}{\partial I_s} = 0 \Rightarrow \frac{\partial \tilde{\mathcal{H}}_s}{\partial I_s} = 0 \tag{19}$$

and

$$\frac{\partial \mathcal{L}_{t-1}}{\partial K_s} = 0 \Rightarrow \frac{\partial \tilde{\mathcal{H}}_{s+1}}{\partial K_s} = R_{t-1,s} q_s \tag{20}$$

Note how the optimality condition with respect to K_s has changed. Equation (20) states that the first derivative of the stock-Hamiltonian must equal the shadow price of capital in the previous period. This is quite intuitive, since differentiating $\tilde{\mathcal{H}}$ with respect to K yields the contribution to current dividends *plus* the value of the capital stock at the end of the period. That is, the marginal effect of additional capital on the value of the firm's activities in periods s, $s+1$, $s+2$, ... In optimum, obviously, this contribution must be equal to the shadow price of capital at the end of the previous period. This way of writing the first order conditions may also be found in Berck and Sydsæter (1993).[2]

[2]More specifically, equation 16.11 on page 86.

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