

# **Empirical rationality in the stock market**

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# Empirical Rationality in the Stock Market

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Rational expectations models make stringent assumptions on the agent's knowledge about the true model. This paper introduces a model in which the rational agent realizes that using a given model involves approximation errors, and adjusts behavior accordingly. If the researcher accounts for this empirical rationality on part of the agent, the resulting empirical model assigns likelihood to the data actually observed, unlike in the unmodified rational expectations case. A Lucas (1978)-type asset pricing model which incorporates empirical rationality is constructed and estimated using U.S. stock data. The equilibrium asset pricing function is seriously affected by the existence of approximation errors and the descriptive properties and normative implications of the model are significantly improved. This suggests that investors do not — and should not — ignore approximation errors.

KEYWORDS: Approximation errors, model uncertainty, estimation of structural models, rational expectations, asset pricing.

## 1. INTRODUCTION

Structural economic models<sup>1</sup> usually provide unique values of the variables representing equilibrium behavior,  $b(x_t)$ , once the values of the state variables,  $x_t$ , are given. This property is troubling in an empirical setting. Since all structural models are simplifications, their implications will be violated by empirical data,  $b_t \neq b(x_t)$ , and two related problems arise. First, there is a strong tradition of modeling rational agents as if they believe the structural model to be an exact description of the economic reality. Rational agents are assumed to ignore the difference between their empirical environment and the theory. Second, since the structural model does not assign likelihood to the empirical data, the researcher

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<sup>1</sup>In this paper, a “structural economic model” is an explicit description of how rational utility maximizing agents determine expectations, decisions, and equilibrium prices from their information about the state of the economy.

is left with a statistically degenerate model where standard maximum likelihood inference is not possible, see Rust (1994). The two problems are obviously related, but have been addressed separately by the literature as *model uncertainty* and the *breaking of statistical degeneracy*. The rest of Section 1 briefly reviews these two strains of the literature, then suggests an *approximation error* approach, providing a unified solution to the two problems.

### 1.1. MODEL UNCERTAINTY

Knight (1921) is the first to address the question of how economic agents react upon uncertainty about the appropriate theoretical model. Variations in the empirical environment are separated into *risk* and *Knightian uncertainty/model uncertainty*<sup>2</sup>. Risk refers to events to which the theoretical model assigns well-defined probabilities, whereas uncertainty refers to events to which no objective probabilities can be assigned. The question of whether or not this distinction should have methodological implications divides the literature in two.

One approach to model uncertainty builds on Ellsberg (1961), who suggests a methodological treatment of model uncertainty outside the traditional paradigm of Savage (1954). Along these lines, Gilboa and Schmeidler (1989) suggest that agents consider a set of possible models and expect the worst model to apply. This *least favorable prior* approach to model uncertainty has been applied by, for instance, Epstein and Wang (1994), Hansen, Sargent, and Tallarini (1999), and Hansen and Sargent (2000). Although this research outside the traditional paradigm of Savage (1954) might have significant descriptive value, Sims (2001) has questioned the normative value of the approach.

Another interpretation of Knight (1921) is found in LeRoy and Singell (1987) and Hirshleifer and Riley (1992, p. 9), see also Arrow (1951), who claim that when no objective probabilities can be assigned to Knightian uncertainty, subjective probabilities are formed and treated as objective probabilities. This interpretation naturally leads to the Bayesian approach to model uncertainty, see, e.g., Draper (1995) and Hansen and Sargent (2000).

Both approaches provide a new optimal behavior,  $\tilde{b}(x_t)$ , and although model uncertainty is admitted for some parts of the model, the maintained hypothesis is that the policy implication is true:

$$(1) \quad b_t = \tilde{b}(x_t).$$

But  $\tilde{b}(x_t)$  is singlevalued for both approaches and (1) is not more likely to apply to empirical data than with  $b(x_t)$  on the right hand side. The statistical degeneracy of the equilibrium implications is not dealt with, and an empirical confrontation must rely on other methods to break the degeneracy.

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<sup>2</sup>The terms *model uncertainty* and *Knightian uncertainty* will be used interchangeably.

## 1.2. BREAKING STATISTICAL DEGENERACY

Following a pragmatic statistical approach, an approximation error term,  $z_t$ , could be added to reflect the shortcomings of the theory:

$$(2) \quad b_t = b(x_t) + z_t.$$

If an appropriate distribution is assumed for  $z_t$ , (2) breaks the statistical degeneracy. However, the approach in (2) is inconsistent from a symmetry point of view. The optimal policy is based on the fact that agents believe  $b_t = b(x_t)$  to be true, which is clearly in conflict with the researcher believing (2).<sup>3</sup> Aware of this problem, the existing literature on maximum likelihood estimation of structural models has insisted that the model is true and used other approaches to circumvent the problem of statistical degeneracy. Mainly, two approaches have been used: The *measurement error* approach and the *unobserved state variable* approach.<sup>4</sup>

The measurement error approach is more confident about the model than the data and assumes that the observed data differ from the *true* empirical values,  $b_t^*$ , by a measurement error term:  $b_t = b_t^* + e_t$ . Since the model is assumed true  $b_t^* = b(x_t)$ , the error term equals the difference between the implications of the theory and the observed data,  $b_t = b(x_t) + e_t$ . If distributional assumptions are made with respect to  $e_t$ , maximum likelihood inference is possible.<sup>5</sup> This approach has mainly been used to estimate macroeconomic models, see Altuğ (1989), Watson (1993), McGrattan (1994), and others.

The unobserved state variable approach recognizes that the singlevalued relationship from state variables to decision and price variables are violated by data. To avoid this direct confrontation, the model is only taken to the data if the state variables, or a subset of these, are unobserved by the researcher. Contrary to the researcher, the agents are assumed to observe the state variables, and their decisions will reflect this knowledge. Using the observed behavior, the state variables are then identified by the researcher,  $x_t = b^{-1}(b_t)$ , and if assumptions are made with respect to the distribution of  $x_t$ , maximum likelihood inference is possible. The approach has been widely used in estimation of structural microeconomic models, see Wolpin (1984), Miller (1984), Pakes (1986), Rust (1987), and others.

Both the measurement error approach and the unobserved state variable approach succeed in breaking the statistical degeneracy while maintaining the hypothesis that the theoretical model is true. The underlying assumptions do not seem valid in general, however. Although the unobserved state variable approach maintains the symmetry assumption with respect to the functional forms of the model, the approach breaks the symmetry with respect to information: The agents

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<sup>3</sup>The discussion here also applies when the behavior is based on the least favorable prior approach or the Bayesian approach to model uncertainty,  $\tilde{b}(x_t)$ . This is not obvious. To see the point, note that although the agent recognizes some kind of model uncertainty while deriving  $\tilde{b}(x_t)$ , this is usually done with the assumption that (1) applies without errors.

<sup>4</sup>Of interest is also the widely used *GMM-approach* to estimation of Euler equations, see Hansen and Singleton (1982), but this approach is applied to models without an explicit description of how expectations are formed, not to structural models as defined above, unless parts of the structure are ignored.

<sup>5</sup>A full measurement error analysis includes measurement errors on the state variables also.

observe more than the researcher. While this asymmetry might be reasonable in a microeconomic setting,<sup>6</sup> it is problematic in a macroeconomic setting. Similarly, measurement errors are not a reasonable explanation in situations where the data quality is high and the empirical fit is poor, as, for instance, in some financial models.

### 1.3. THE APPROXIMATION ERROR APPROACH

By allowing approximation errors, this paper treats model uncertainty and statistical degeneracy in a unified setting. The hope of finding the *true* structural model is abandoned. Instead, a *useful* structural model is considered. Neither the agent nor the researcher believes that the useful structural model is exactly true, but recognize approximation errors as the reason for the difference between theory and data. The result is an integration of (1) and (2).

The approximation error approach differs from the traditional approaches to breaking the statistical degeneracy since the decisions of the agents are affected by model uncertainty,  $\tilde{b}(x_t) \neq b(x_t)$ . The approach also differs from traditional approaches to model uncertainty, since the implications of the theory, even when approximation errors are taken into account by the agents, can differ from the empirical data:

$$(3) \quad b_t = \tilde{b}(x_t) + z_t.$$

Following Knight (1921), the  $\tilde{b}(x_t)$ -term should be interpreted as the risk-part of the empirical variation in  $b_t$ , that is, the part to which well-defined objective probabilities are assigned. The approximation error term,  $z_t$ , represents Knightian uncertainty/model uncertainty, since this part of the variation is not accounted for by the structural theoretical model.

This paper is kept in the traditional normative framework of Savage (1954) by following LeRoy and Singell (1987): Subjective beliefs with respect to the distribution of  $z_t$  are formed and treated as objective probabilities. To close the model with respect to expectations, the standard symmetry assumption of the rational expectation tradition is imposed: If the researcher believes (3), given a stochastic distribution of  $z_t$ , the agents should determine  $\tilde{b}(x_t)$  believing (3) and the same distributional assumptions as the researcher. If the beliefs are empirically relevant, this approach will break the statistical degeneracy and allow for standard maximum likelihood inference.

It might not be obvious why  $b_t$  is chosen by agents when  $\tilde{b}(x_t)$  is recommended by the structural model. The problem is especially troubling for micro models. The argument is that agents consider the structural model only as a useful tool, not the true description of the economic environment. The individual agent recognizes that events not described by the structural model do take place and make the agent deliberately decide differently from what is implied by the model.<sup>7</sup>

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<sup>6</sup>An obvious exception is when the researcher and the agent happen to be the same person.

<sup>7</sup>The optimization error approach to statistical degeneracy assumes that the difference between the optimal and the actual decisions are unintended by the agents, see Rust (1994). The approach, which has received little attention in the literature, can be seen as a special case of the approxi-

The approximation error approach has several advantages. First, the approach succeeds in introducing model uncertainty without giving up the rationality of Savage's normative framework, thereby avoiding the criticism of Sims (2001). Of course, from a descriptive point of view, this might be less of an advantage. The approach is not capable of addressing observations like the Paradox of Ellsberg (1961). Still, the approximation error approach does not exclude the introduction of Bayesian learning with respect to the most useful model, or that the agents form expectations according to the least favorable prior approach.

Second, unlike both measurement errors and unobserved state variables, it is hard to imagine situations where approximation errors are not a reasonable explanation of a deviation between theory and data, and the approach may, of course, be combined with measurement errors or unobserved state variables when this is appropriate.

Finally, the approximation error approach offers technical advantages. The measurement error approach cannot be applied to nonlinear specifications, except in special cases, see Rust (1994). Moreover, to apply the unobserved state variable approach, an invertibility condition must be satisfied, see Pakes (1994). As shown below, such restrictions are not imposed by the approximation error approach.

The rest of the paper is organized as follows. Section 2 describes a simple version of the Lucas (1978) asset pricing model and shows that the model is statistically degenerate and subject to approximation errors. Section 3 adds approximation errors to the model and re-optimizes with respect to the rational investor's investment problem. Section 4 estimates the model using U.S. stock data and shows how both the descriptive and the normative properties of the model are significantly improved, once approximation errors are taken into account. From a descriptive point of view, the *equity premium puzzle* of Mehra and Prescott (1985), as well as the *stock market volatility puzzle*, see Shiller (1981) and Grossman and Shiller (1981), are addressed. From a normative point of view, the model predicts expected excess returns and the risk involved in exploiting these much more accurately than the traditional model without approximation errors.

## 2. A THEORETICAL ASSET PRICING MODEL

Assume that a Lucas (1978)-type asset pricing model is a useful theoretical model for the situation at hand. At time  $t$  a representative investor, or equivalently a number of identical investors, receives an endowment,  $e_t$ , which can be used for consumption,  $c_t$ , to gain utility,  $u(c_t)$ , or it can be used for investments in a financial asset which pays stochastic dividends  $d$  in the future. To maximize the expected infinite horizon utility at an arbitrary point in time, say  $t = 0$ , the investor solves the following problem:

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mation error approach except that  $z_t$  is given a specific interpretation. Hence, the optimization error approach should be implemented similar to the approximation error approach.

$$(4) \quad \max_w E_0 \sum_{t=0}^{\infty} \beta^t u(c_t),$$

*s.t.*

$$c_t = e_t + d_t w_t + p_t(w_t - w_{t+1}),$$

where  $d_0$  and  $c_0$  are given,  $w_t$  is the investor's holding of the financial asset,  $p_t$  is the price of the asset, and  $0 < \beta < 1$  is a constant discount factor.  $E_0$  denotes expectations conditioned at information available at  $t = 0$ . The utility function is assumed to be a standard *constant relative risk aversion*-type:

$$u(c_t) = \frac{c_t^{1-\gamma} - 1}{1-\gamma},$$

where  $\gamma > 0$ . To simplify,  $w$  is assumed to represent excess demand relative to a fixed supply. Therefore,  $w_t = 0$  and  $c_t = e_t$  in equilibrium. In this model,  $e_t$  (and hence  $c_t$ ) and  $d_t$  represent payoffs to equilibrium asset holdings and excess holdings, respectively. Consumption (or, equivalently, endowments) and dividends are assumed to follow stationary exogenous Markov processes. In the sequel, both will affect the equilibrium price: Consumption by determining marginal utility, and dividends by determining asset payoffs. Two alternative autoregressive specifications are considered in this paper: One specification with additive shocks,

$$(5.A) \quad \begin{aligned} d_t &= (1 - \delta) + \delta d_{t-1} + \epsilon_t^d, \\ c_t &= (1 - \theta) + \theta c_{t-1} + \epsilon_t^c, \end{aligned}$$

and one with multiplicative shocks,

$$(5.M) \quad \begin{aligned} d_t &= d_{t-1}^\delta \exp(\epsilon_t^d), \\ c_t &= c_{t-1}^\theta \exp(\epsilon_t^c), \end{aligned}$$

where  $|\delta| < 1$ ,  $|\theta| < 1$ , and  $\epsilon_t = [\epsilon_t^d, \epsilon_t^c]$  are independent normally distributed shocks with mean vector  $\mu$  and covariance matrix  $\Omega$ . Although not indicated by the notation,  $\mu$  and  $\Omega$  differ between (5.A) and (5.M). For instance,  $\mu$  is set to zero in the additive case, but slightly lower in the multiplicative case, to insure that the unconditional expected values of  $d_t$  and  $c_t$  equal one, see Appendix A.<sup>8</sup> Each of the two specifications above has advantages over the other. The multiplicative specification in (5.M) offers closed form solutions for the equilibrium price, which the additive specification in (5.A) does not. However, Section 3 shows that the additive specification provides the most intuitive interpretation of the effects of approximation errors.

To solve for the equilibrium price at time 0, consider the Euler equation obtained from using the first order condition with respect to  $w_1$  and the equilibrium condition

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<sup>8</sup>The additive specification assigns likelihood to negative values although negative consumption is not allowed when  $\gamma \geq 1$ . However, given realistic parameter values, the probability of negative consumption is negligible. Moreover, the empirical analysis suggest that  $\gamma$  is not significantly greater than one.

$w_t = 0$ ,<sup>9</sup>

$$(6) \quad p(d_0, c_0) = E_0 \tilde{\beta}(d_1 + p_1), \quad \tilde{\beta} \equiv \beta \frac{u'(c_1)}{u'(c_0)}.$$

Due to the Markov property,  $d_0$  and  $c_0$  summarize all available information about the state of the economy at  $t = 0$ . To determine the equilibrium price,  $p(d_0, c_0)$ , only the investor's expectations with respect to  $p_1$  are left to be specified. The usual way to proceed is to follow the rational expectation paradigm initiated by Muth (1961) and substitute  $p_1$  with  $p(d_1, c_1)$  to get a functional equation for the price function:

$$(7) \quad p^T(d_0, c_0) = E_0 \tilde{\beta}(d_1 + p^T(d_1, c_1)).$$

Here,  $p^T$  denotes the price function derived based on the assumption that rational agents expect future prices to be perfectly accounted for by the theoretical model. This is the key assumption which is modified under the approximation error approach in Section 3.

Following the conventional procedure,  $p^T(d_1, c_1)$  in (7) might be replaced by the right-hand side of the Euler equation derived from the first order condition with respect to  $w_2$ , i.e., (7) led once. This leaves  $p^T(d_0, c_0)$  as a function of  $p_2$  or, when expectations are formed using the theoretical model, as a function of  $p^T(d_2, c_2)$ . Applying this substitution repeatedly (and ruling out bobbles etc.) gives

$$(8) \quad p^T(d_0, c_0) = E_0 \sum_{t=1}^{\infty} \tilde{\beta}^t d_t, \quad \tilde{\beta}^t \equiv \beta^t \frac{u'(c_t)}{u'(c_0)}.$$

The price equals the expected sum of dividends discounted by a stochastic discount factor  $\tilde{\beta}^t$  depending on the marginal utility of consumption. If the multiplicative specification in (5.M) is assumed, the expected value in (8) has a closed form solution:

$$(9) \quad p^T(d_0, c_0) = \sum_{t=1}^{\infty} \beta^t c_0^{-\gamma(\alpha^t - 1)} d_0^{\delta^t} \Lambda_t,$$

where  $\Lambda_t$  depend only on parameters and  $t$ , see Appendix A. Closed form solutions are not available when the additive specification in (5.A) is assumed, and numerical procedures are used below to determine the equilibrium price in this case, see Appendix B.

The equilibrium price expression in (9) clearly illustrates the statistical degeneracy common to most structural models. If empirical values for  $d_0$  and  $c_0$  are observed,  $p^T(d_0, c_0)$  can easily be calculated for fixed parameters. Since the result

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<sup>9</sup>In order to simplify the exposition, the agents are assumed to know the true parameter values of the model, whereas the researcher must estimate them, see Section 4. To reestablish the symmetry, a Bayesian approach could be adopted with respect to both the researcher's and the agents' beliefs about the parameter values.



is a single real number, it is very unlikely that this equilibrium price will ever match the empirical counterpart,  $p_0$ . Varying the parameters or choosing other functional forms might bring the model to match the data for some observations, but in general

$$(10) \quad p_t \neq p^T(d_t, c_t)$$

for (almost) all observations.<sup>10</sup> Since the model does not assign likelihood to situations like (10), the model is statistically degenerated and maximum likelihood inference not possible.

Measurement errors can hardly explain the empirical deviation. As shown empirically in Section 4, the gap is simply too big. Regarding the unobserved state variable approach, it is hard to deny that more state variables could improve the model, but it is equally hard to believe that these should be known to the representative agent and not to a single researcher in the academic community, cf. the *stock market volatility puzzle*. In this case, approximation errors seem like a more promising approach to break the statistical degeneracy.

### 3. AN EMPIRICAL ASSET PRICING MODEL

This section introduces an empirical model that is based on the idea of empirical rationality. Before proceeding, some useful terminology is introduced. Let a concrete empirical data sample be given. First, a model that assigns no likelihood to the empirical data is henceforth referred to as a *theoretical model* with respect to the empirical data. Expectations formed using a theoretical model are *theoretical expectations*. Behavior, which is optimal according to a theoretical model and the corresponding theoretical expectations, reflects *theoretical rationality*.

Notice that the asset pricing model described in Section 2 is clearly theoretical and since  $p_t = p^T(d_t, c_t)$  is expected to apply for future prices, the expectations are theoretical too. The equilibrium price in (8) therefore reflects theoretical rationality.

Secondly, a model that does assign likelihood to the empirical data is an *empirical model*. Expectations formed using an empirical model are *empirical expectations*. Behavior, which is optimal according to an empirical model and the corresponding empirical expectations, reflects *empirical rationality*.

#### 3.1. EMPIRICAL RATIONALITY: ADDITIVE SPECIFICATION

Assume, as the researcher, that the model in (4) with the additive specifications in (5.A) is a useful theoretical model. Yet, the implication of the model (here the equilibrium price  $p^E$ ), is assumed to differ from the empirical asset price  $p_t$  by an approximation error term  $z$ ,

$$(11) \quad p_t = p^E + z_t,$$

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<sup>10</sup>The trivial exception is the case where the number of free parameters equals or exceeds the number of observations multiplied with the number of equilibrium implications of the model.

where  $p^E$  is the equilibrium price of the empirical model which is to be determined below. Parallel to the interpretation of (3),  $p^E$  is the implication of the structural model whereas  $z_t$  reflects Knightian uncertainty/model uncertainty. Let the subjective beliefs of both the researcher and the investor be described by

$$(12) \quad z_t = \zeta z_{t-1} + \epsilon_t^z,$$

where  $|\zeta| < 1$  and  $\epsilon_t = [\epsilon_t^d \ \epsilon_t^c \ \epsilon_t^z]^\top$  is independent across time and jointly normally distributed with mean vector  $\mu = 0$  and covariance matrix  $\Omega$ . Since likelihood is assigned to all real values of  $z_t$ , the model is empirical as long as  $p_t$  and  $p^E$  are finite. Finally, assume that also the representative investor is aware of the approximation errors and use (11) and (12) to form expectations about future prices. Thus, empirical expectations are formed.

In order to determine the equilibrium price based on empirical rationality, notice that the Euler equation in (6) is still valid. However, when expectations with respect to next period's price are formed,  $p_1$  should be replaced by  $p^E + z_t$  instead of by the theoretical price  $p^T$ . Therefore, (7) is replaced by

$$(13) \quad p^E(d_0, c_0, z_0) = E_0 \tilde{\beta}(d_1 + p^E(d_1, c_1, z_1) + z_1).$$

Applying the usual repeated substitution procedure, but now using (11), gives

$$(14) \quad \begin{aligned} p^E(d_0, c_0, z_0) &= E_0 \sum_{t=1}^{\infty} \tilde{\beta}^t (d_t + z_t) \\ &= p^T(d_0, c_0) + E_0 \sum_{t=1}^{\infty} \tilde{\beta}^t z_t. \end{aligned}$$

Notice that  $p^E$  coincides with  $p^T$  if approximation errors are absent. The effect of the approximation errors under empirical rationality is naturally decomposed into a *predictability effect*,  $\Delta_t$ , and an *uncertainty premium*,  $\pi_t^z$ :

$$(15) \quad p^E(d_0, c_0, z_0) - p^T(d_0, c_0) = \underbrace{E_0 \sum_{t=1}^{\infty} \tilde{\beta}^t E_0(z_t)}_{\Delta_0} + \underbrace{E_0 \sum_{t=1}^{\infty} \tilde{\beta}^t (z_t - E_0(z_t))}_{\pi_0^z}.$$

The predictability effect arises when  $\zeta \neq 0$ . Assume, for instance, that the approximation errors are positively autocorrelated,  $\zeta > 0$ , and assume that the investor observes a higher price than predicted by the equilibrium price,  $z_0 > 0$ . Then, positive approximation errors should also be expected in the next period,  $E_0 z_1 = \zeta z_0 > 0$ . The effect on the current equilibrium price is then illustrated by the Euler equation in (13) with  $z_1$  replaced with  $E_0 z_1 = \zeta z_0$ . Since the investor predicts empirical prices above next period's equilibrium price, the current equilibrium price is increased by  $E_0 \tilde{\beta} \zeta z_0$ . However, next periods equilibrium price,  $p^E(d_1, c_1, z_1)$ , and all other future prices are also affected by the fact that all future approximation errors are expected to be positive. The empirically rational investor foresees the increase in future equilibrium prices, and hence the current equilibrium

price is increased even further by  $E_0 \sum_{t=2}^{\infty} \tilde{\beta}^t \zeta^t z_0$ . The sum of this derived effect and the direct effect equals  $\Delta_0$  in (15).

Notice that in the case of constant consumption ( $c_t = 1$ ) the predictability simplifies to

$$(16) \quad \Delta_0 = \frac{\beta\zeta}{1 - \beta\zeta} z_0.$$

Thus for  $\zeta$ - and  $\beta$ -values close to one, the predictability effect may be considerably larger than the approximation error itself.

The uncertainty premium,  $\pi_t^z$ , arises when the unexpected variation in the approximation errors is correlated with the consumption variation in the stochastic discount factor  $\tilde{\beta}^t$ . This happens when  $\text{cov}(\epsilon_t^z, \epsilon_t^c)$  is nonzero. This definition of  $\pi_t^z$  is similar to the definition of the traditional *risk premium*,  $\pi_t^d$ , which captures the uncertainty of payoff in the theoretical model:<sup>11</sup>

$$\pi_0^d = E_0 \sum_{t=1}^{\infty} \tilde{\beta}^t (d_t - E_0(d_t)).$$

Besides premiums generated by variation in  $d_t$  and  $z_t$ , the wish to smooth consumption, even when  $d_t$  and  $z_t$  are deterministic, generates a *consumption risk premium*:

$$\pi_0^c = E_0 \sum_{t=1}^{\infty} \beta^t \frac{u'(c_t) - u'(E_0(c_t))}{u'(c_0)} (E_0(d_t) + E_0(z_t)).$$

Notice that for the additive specification (5.A) and (11),  $\pi_t^d + \pi_t^c + \pi_t^z$  equals the difference between the equilibrium price and the *certainty equivalent* counterpart. To see this, note that the certainty equivalent equilibrium price is given by  $\sum_{t=1}^{\infty} \beta^t \frac{u'(E_0(c_t))}{u'(c_0)} (E_0(d_t) + E_0(z_t))$ . Section 4 estimates and compares the three premiums in order to assess the importance of  $\pi_t^z$ .

Notice also that if the approximation errors are white noise,<sup>12</sup> the approximation errors will not affect the equilibrium price. In this case, the simple empirical model  $p_t = p^T(d_0, c_0) + \epsilon_t^z$  would in fact be consistent with empirical rationality.

### 3.2. EMPIRICAL RATIONALITY: MULTIPLICATIVE SPECIFICATION

Again, let (4) be a relevant theoretical asset pricing model, but let  $d$  and  $c$  follow the multiplicative processes defined in (5.M). Moreover, assume that the beliefs with respect to approximation errors are described by

$$(17) \quad p_t = p^E z_t,$$

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<sup>11</sup>The distinction between uncertainty premiums and risk premiums might appear subtle in the setting above. It is hard to argue that the probabilities assigned to the evolution of  $d_t$  and  $c_t$  by (5.A) and (5.M) are more objective than those assigned to the evolution of  $z_t$ . Nevertheless, the usual term *risk* is maintained.

<sup>12</sup>In this setting,  $\zeta = \text{cov}(\epsilon_t^z, \epsilon_t^d) = \text{cov}(\epsilon_t^z, \epsilon_t^c) = 0$ .

where the specification of the  $z$ -process corresponds to the  $d$  and  $c$  processes:

$$(18) \quad z_t = z_{t-1}^\zeta \exp \epsilon_t^z.$$

As before,  $|\zeta| < 1$  and  $\epsilon_t = [\epsilon_t^d \ \epsilon_t^c \ \epsilon_t^z]^\top$  is time-independent jointly normally distributed. Finally, assume that the representative investor forms empirical expectations according to (17) and (18). Contrary to the additive model, the model described by (4), (5.M), (17), and (18) does not assign likelihood to negative values of  $d$ ,  $c$ , and  $z$ . However, the model is still empirical if  $p_t$  and  $p^E$  take finite positive values.

The Euler equation in (6) is still valid. However, once the uncertainty augmented expectation equation in (17) is used to substitute for  $p_1$ , the relevant Euler equation becomes

$$p^E(d_0, c_0, z_0) = E_0 \left[ \tilde{\beta} (d_1 + p^E(d_1, c_1, z_1) z_1) \right].$$

Repeated substitution using (17) yields

$$p^E(d_0, c_0, z_0) = E_0 \sum_{t=1}^{\infty} \tilde{\beta}^t d_t z_{t-1} z_{t-2} \cdots z_1.$$

Notice that  $p^E$  coincides with  $p^T$  if the approximation error shocks  $\epsilon^z$  vanish. Even with approximation errors, the multiplicative empirical rationality model offers a closed form solution for the equilibrium asset pricing function,

$$(19) \quad p^E(d_0, c_0, z_0) = \sum_{t=1}^{\infty} \beta^t c_0^{-\gamma(\theta^t-1)} d_0^{\delta^t} z_0^{\frac{\zeta-\zeta^t}{1-\zeta}} \Psi_t,$$

where  $\Psi_t$  depends on parameters and  $t$ , only, see Appendix A. The  $z_0$ -term in (19) clearly indicates a predictability effect when  $\zeta \neq 0$ . Nevertheless, the effect of the approximation errors under empirical rationality is not decomposed as easily as with additive specification. The definitions of the different kinds of risk and uncertainty premiums may be found in Appendix A, together with their closed form expressions.

Although the approximation error specification in (17) and (18) is stationary, the effect on the equilibrium price can be dramatic. Appendix A shows that  $p^E$  is infinite when

$$(20) \quad \mu_z > -\frac{\sigma_z^2}{2(1-\zeta)} \equiv \bar{\mu}_z,$$

where  $\mu_z$  and  $\sigma_z^2$  denote the mean and variance of  $\epsilon^z$ . Even  $\mu_z = 0$  results in an infinite equilibrium price. For  $\mu_z \leq \bar{\mu}_z$ , however,  $p^E$  is finite and well-behaved. Thus, the equilibrium price displays a serious discontinuity in parameters. In the empirical analysis,  $\mu_z$  is determined by assuming (20) to apply with equality, this being the closest possible value to that implied by the approach used for estimating the  $c_t$  and  $d_t$  processes (see Appendix A) which still results in finite prices.

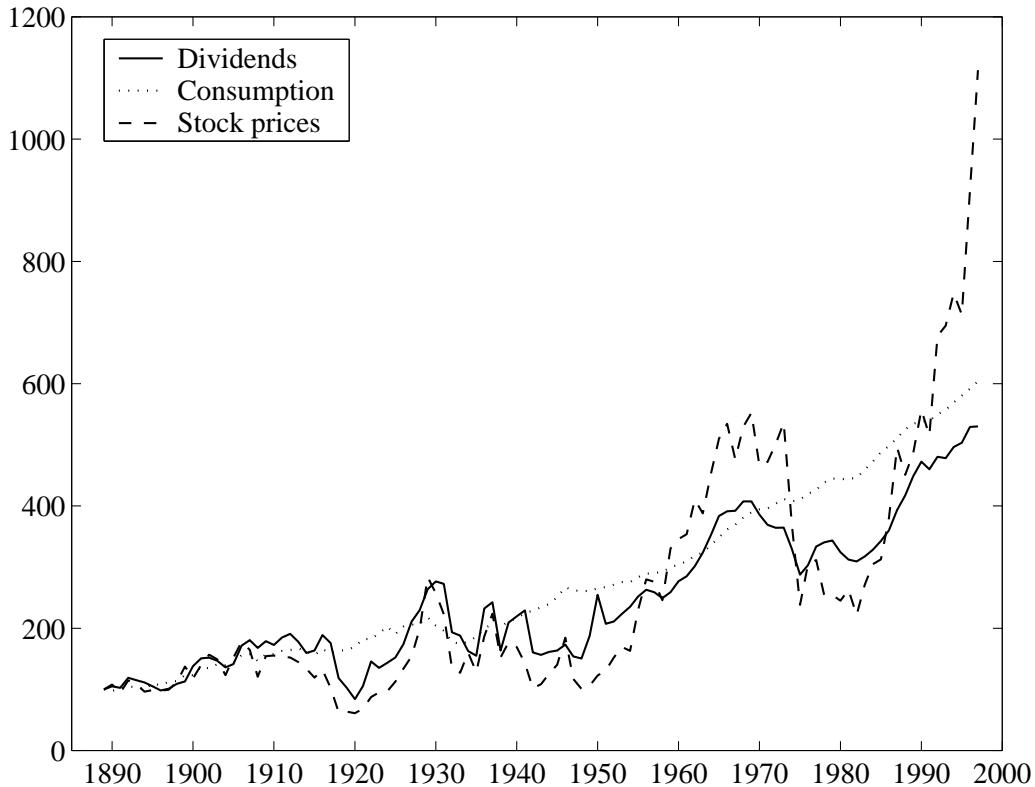


FIGURE 1: The empirical series, real values

The dividends and the stock prices refer to the S&P Composite Stock Price Index provided by Robert J. Shiller from his Yale-homepage. Consumption is real per capita consumption of non-durables and services. Annual data, 1889–1997. (1889 = 100)

## 4. ESTIMATION

### 4.1. DATA

The data used for the empirical analysis are real annual dividends and stock prices for the S&P Composite Stock Price Index and real per capita consumption series used in Shiller (1989). The data have been updated (dividends and prices) and made publicly available by Robert Shiller via his homepage. The consumption series have been updated with data from the Bureau of Economic Analysis. The sample period is 1889–1997. Figure 1 shows real value indexes for all three variables of the model. In order to work with a finite state space, the series are detrended with a constant growth rate of 1.60% per year, which is the growth rate of per capita consumption during the estimation period. The result is seen in Figure 2. Finally, the mean of the detrended  $d$  and  $c$  series are normalized to one and the price series is transformed accordingly to maintain the average price-dividend ratio of 22.8 from the raw data.

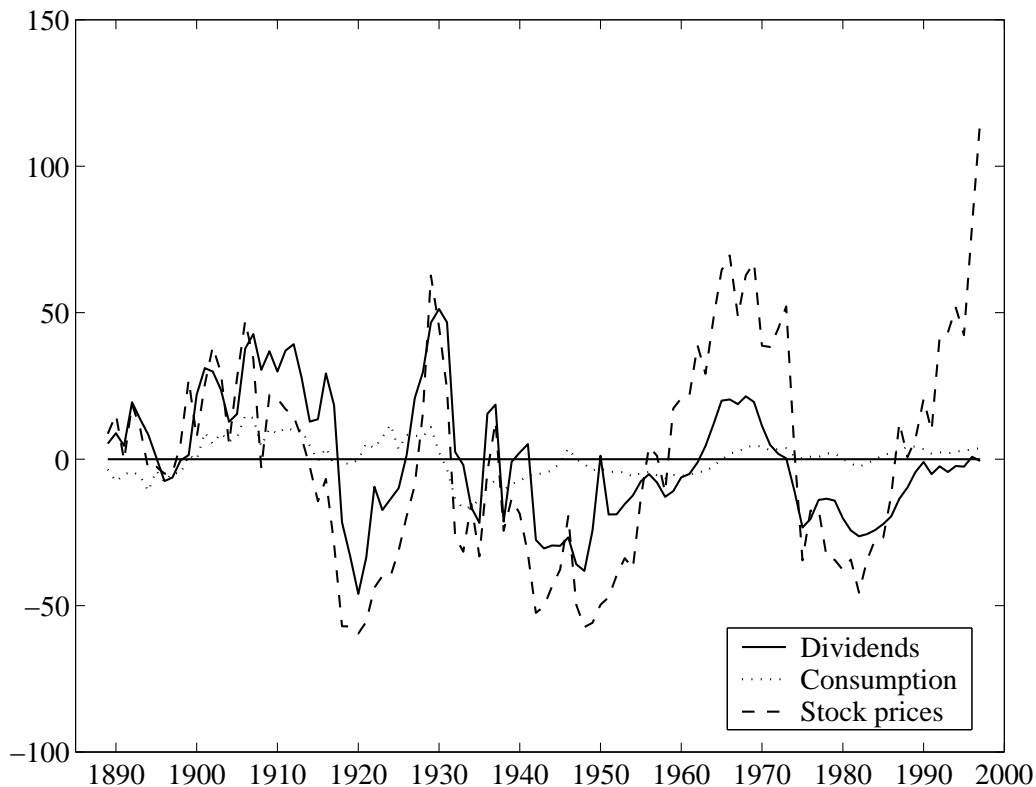


FIGURE 2: Detrended series (% from mean)

#### 4.2. ESTIMATION OF THE EXOGENOUS PROCESSES $d$ AND $c$

Since the  $d$  and  $c$  processes are exogenous to the model, it is possible to estimate their parameters ( $\delta$ ,  $\gamma$ , and the relevant elements of  $\Omega$ ) independent of the asset pricing model in (4). Table 1 shows maximum likelihood estimates of the parameters in (5.A) and (5.M). The choice of specification seems to play a minor role for the estimates. Both dividends and consumption are significantly serially correlated and nearly 75% of the total variation is explained by these simple specifications. For both specifications, the residuals display significantly more kurtosis than the normal distribution, whereas the skewness estimates are more consistent with the normality assumption.

The estimates in Table 1 will be taken as given when solving for the pricing function and estimating the other parameters of the model.

#### 4.3. THE MODELS

In order to assess the effect of empirical rationality, the asset pricing model is estimated in a theoretical rationality version as well as an empirical rationality version. Both versions are estimated with additive and multiplicative specifications. Consider first the two models based on empirical rationality analyzed in details in Section 3:

TABLE 1  
ESTIMATES OF  $d$  AND  $c$  PROCESS PARAMETERS

Specification		$\delta/\theta$	$\sigma_d/\sigma_c$	$\rho_{d,c}$	$R^2$	<i>Normality</i> <i>P-value</i>	DW
Additive	d	0.8740 (0.0485)	0.1145 (0.0078)	0.4063 (0.0816)	0.7087	38.5176 0.0000	1.7007
	c	0.8341 (0.0455)	0.0317 (0.0022)		0.7444	8.7012 0.0129	2.0768
Multiplic.	d	0.8652 (0.0499)	0.1179 (0.0080)	0.3611 (0.0850)	0.7090	33.4978 0.0000	1.6897
	c	0.8437 (0.0453)	0.0314 (0.0021)		0.7456	13.3207 0.0013	2.1021

*Notes:*  $\sigma_d$  and  $\sigma_c$  are standard deviations of  $\epsilon^d$  and  $\epsilon^c$ .  $\rho_{d,c}$  is the correlation. Figures in brackets are standard deviations based on the hessian of the loglikelihood function. The normality test is a  $\chi(2)$ -distributed Jarque–Bera joint skewness and kurtosis test. DW is the Durbin–Watson test statistic for autocorrelation.

$$M_A^E \quad p_t = p^E(d_t, c_t, z_t) + z_t,$$

$$M_M^E \quad p_t = p^E(d_t, c_t, z_t)z_t,$$

where the subscripts of  $M_A^E$  and  $M_M^E$  refer to additive and multiplicative specification, respectively, and the superscript refer to empirical rationality. Their counterparts are

$$M_A^T \quad p_t = p^T(d_t, c_t) + z_t,$$

$$M_M^T \quad p_t = p^T(d_t, c_t)z_t,$$

where the superscript indicates that the equilibrium price is based on theoretical rationality. Thus, even though the researcher believes  $M_A^T$  and  $M_M^T$  (which are in fact empirical models, since the presence of  $z_t$  breaks the statistical degeneracy and allows assigning likelihood to data), the investor ignores  $z_t$  when the equilibrium price is determined.

#### 4.4. THE LIKELIHOOD FUNCTION

The likelihood of the price series  $p_t$ ,  $t = \{2, \dots, T\}$ , is conditioned on the first observation  $p_1$  and the free parameters  $\omega = (\gamma, \beta, \zeta, \sigma_z, \rho_{d,z}, \rho_{c,z})$ . As mentioned, the observations of  $d_t$  and  $c_t$ , the parameters of Table 1 and the implied values of  $\epsilon_d$  and  $\epsilon_c$  are exogenous by assumption. The loglikelihood of the price series is then given by

$$\begin{aligned}\ell(p_T, p_{T-1}, \dots, p_2 | p_1, \omega) &= \sum_{t=2}^T \ell(p_t | p_{t-1}, \omega) \\ &= \sum_{t=2}^T \left( \frac{\partial M}{\partial z_t} \frac{\partial z_t}{\partial \epsilon_t^z} \right)^{-1} \ell(\epsilon_t^z | \omega),\end{aligned}$$

where  $M \in \{M_A^T, M_M^T, M_A^E, M_M^E\}$  and the Jacobian  $\frac{\partial M}{\partial z} \frac{\partial z}{\partial \epsilon^z}$  is used for change of measure from  $\epsilon^z$  to  $p$ .

#### 4.5. PARAMETER ESTIMATES

Panel A of Table 2 shows the estimation results. Notice that  $\gamma$  is imprecisely estimated, and, based on a 95% likelihood ratio test, log-utility ( $\gamma = 1$ ) cannot be rejected for any of the four models.

Panel B of Table 2 shows the estimation results with the log-utility condition imposed. The normality of  $z_t$  is now accepted only at a 1% significance level for the multiplicative specification. In return, the precision of all estimates as well as the  $R^2$ -values are improved compared to Panel A. Since the restriction also facilitates a more direct comparison of risk and uncertainty premiums across models, log-utility is imposed in the analysis below.

Except for  $\sigma_z$ , the parameter estimates in Panel B are constant across models. The approximation errors show high autocorrelation, with estimates of  $\zeta$  around 0.9 and strong significance. Thus, the predictability effect,  $\Delta$ , of approximation errors is significant. The correlations between the shocks,  $\rho_{d,z}$  and  $\rho_{c,z}$ , are significantly positive, at approximately 0.5 and 0.35, respectively. The uncertainty premium,  $\pi^z$ , is therefore significant. As the estimates in Panel A show, this conclusion depends on the log-utility restriction, however. The discount factor estimates are between 0.96 and 0.965<sup>13</sup>. Keeping the rationality type (empirical or theoretical) fixed, the ratio of the  $\sigma_z$ -estimates for the additive and the multiplicative specification is approximately equal to the empirical price/dividend ratio of 22.8.

An important result in Table 2 is the dramatic decrease in the size of approximation errors when empirical rationality is introduced. As a result, the estimates of  $\sigma_z$  are reduced by 85-90%. The reduction is accompanied by an equally dramatic increase in the  $R^2$ -values. Both changes are mainly due to a major predictability effect.

#### 4.6. THE PREDICTABILITY EFFECT

The reduction in approximation errors caused by empirical rationality is clearly illustrated in Figure 3. The empirical prices are shown together with the equilibrium prices for both the theoretical and the empirical rationality model with additive shocks. Obviously,  $p^T$  explains very little of the empirical price variation. This *stock market excess volatility* is well documented by Shiller (1981) and Grossman and Shiller (1981).

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<sup>13</sup>Notice that in order to relate to yearly returns the  $\beta$ -estimates should be corrected for the growth rate of 1.6%.



TABLE 2  
ESTIMATION RESULTS

	$\beta$	$\gamma$	$\zeta$	$\sigma_z$	$\rho_{d,z}$	$\rho_{c,z}$	Normality <i>P-value</i>	DW	$R^2$	$\ell$
Panel A: All parameters free										
$M_A^T$	0.9637 (0.0077)	2.3458 (1.1668)	0.9417 (0.0430)	3.2620 (0.2206)	0.3392 (0.1229)	0.0255 (0.2742)	0.5857 0.7462	1.8964	0.0422	-273.41
$M_M^T$	0.9600 (0.0057)	4.3083 (4.0818)	0.9098 (0.0364)	0.1499 (0.0258)	0.3552 (0.3121)	-0.2234 (0.6669)	4.2563 0.1191	1.9832	0.0548	-262.76
$M_A^E$	0.9615 (0.0071)	3.4303 (1.5484)	0.9288 (0.0413)	0.3399 (0.1454)	0.3010 (0.1381)	-0.1052 (0.2938)	0.1486 0.9284	1.9771	0.9904	-270.95
$M_M^E$	0.9642 (0.0060)	5.7121 (6.9891)	0.8984 (0.0383)	0.0224 (0.0155)	0.2449 (0.5139)	-0.4293 (0.9053)	3.0585 0.2167	2.0246	0.9806	-262.83
Panel B: $\gamma = 1$ imposed										
$M_A^T$	0.9639 (0.0074)	1	0.9358 (0.0465)	3.4270 (0.2383)	0.4254 (0.0777)	0.3120 (0.0928)	0.6341 0.7283	1.8346	0.0490	-273.94
$M_M^T$	0.9603 (0.0051)	1	0.9020 (0.0371)	0.1549 (0.0101)	0.5368 (0.0646)	0.3291 (0.0740)	7.8357 0.0199	1.8175	0.1998	-263.17
$M_A^E$	0.9642 (0.0072)	1	0.9360 (0.0444)	0.3236 (0.1632)	0.4366 (0.0740)	0.3340 (0.0798)	0.2912 0.8645	1.8577	0.9912	-272.74
$M_M^E$	0.9645 (0.0055)	1	0.8937 (0.0345)	0.0221 (0.0055)	0.5287 (0.0638)	0.3293 (0.0744)	8.7039 0.0129	1.7855	0.9860	-263.34

*Notes:* Figures in brackets are standard deviations based on the hessian of the loglikelihood function. The normality test is the  $\chi(2)$ -distributed Jarque–Bera joint skewness and kurtosis test. DW is the Durbin–Watson test statistic for autocorrelation. The  $R^2$ -figures are based on a simple additive specification:  $p_t = p^E(d_t, c_t, z_t) + e_t$ , where the only explanatory effect of  $z$  is through  $p^E$ .  $\ell$  is the value of the loglikelihood function.

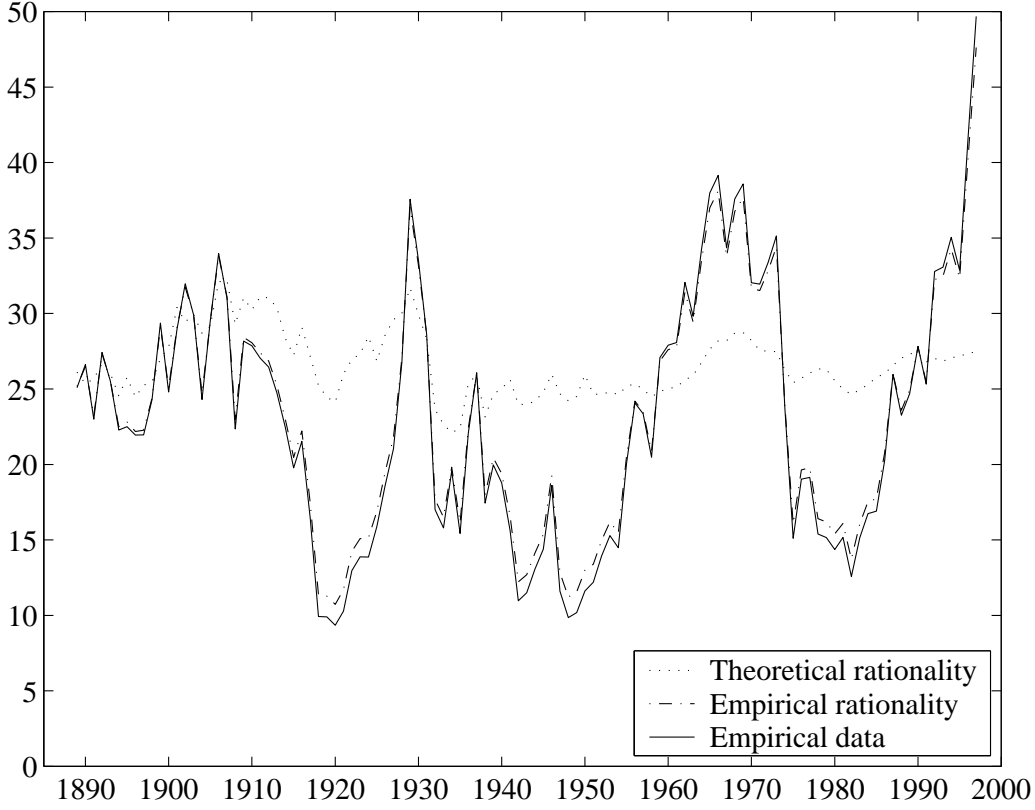


FIGURE 3: Empirical price and model prices.  
The equilibrium prices are based at  $M_A^T$  and  $M_A^E$  and the relevant parameter values of Panel B of Table 2.

The empirical fit of  $p^E$  is obviously much better. The reason for this dramatic change is explained by the high persistence of the approximation errors, combined with the predictability effect. To simplify the analysis, ignore risk premiums for the moment and assume that future values of  $z$  and  $c$  equal their expected values. Assuming additive specification and using (14) and (16), the model can be decomposed as:

$$\begin{aligned}
(21) \quad p_t &= p^E(d_t, c_t, z_t) + z_t \\
&= p^T(d_t, c_t) + \frac{\beta\zeta}{1 - \beta\zeta} z_t + z_t \\
&\approx p^T(d_t, c_t) + 9z_t + z_t, \quad \text{when } \beta \approx 0.96 \text{ and } \zeta \approx 0.94.
\end{aligned}$$

The last line of (21) shows that when the empirical price differs from the one predicted by the theoretical model, the equilibrium price of the empirical model reacts very strongly. Thus, 10% of the approximation error for the theoretical model now explains the empirical variation. The remaining 90% are accounted for by a change in the equilibrium price, due to the predictability of future approximation errors.

TABLE 3

RISK AND UNCERTAINTY PREMIUMS: EQUILIBRIUM PRICE CHANGES IN %

Model		Risk		Uncertainty
		$\pi^c$	$\pi^d$	$\pi^z$
$M_A^T$	Steady State	0.306	-0.500	-
	Data	0.306 (0.006)	-0.502 (0.022)	-
$M_M^T$	Steady State	0.313	-0.448	-
	Data	0.313 (0.003)	-0.448 (0.004)	-
$M_A^E$	Steady State	0.312	-0.508	-1.431
	Data	0.323 (0.020)	-0.643 (0.225)	-1.811 (0.632)
$M_M^E$	Steady State	0.316	-0.440	-0.352
	Data	0.312 (0.006)	-0.434 (0.008)	-0.341 (0.017)

*Notes:* “Steady state” is the “deterministic steady state”:  $(d, c, z) = (0, 0, 0)/(1, 1, 1)$  for the additive/multiplicative specification. “Data” is average effects implied by the models for the empirical data-series, 1889-1997. Figures in brackets are standard deviations. All figures are based on the estimates in Table 2, Panel B.

This explains the 90% drop in the estimated  $\sigma_z$ -values in Table 2. With a 85% drop in the  $\sigma_z$  estimate, the results for the multiplicative specification are similar to those of the additive specification.

Adding approximation errors does not explain why the empirical stock price deviates from the theoretical one in the first place, but the model does show that a significant part of the observed deviation might actually be a rational response to a smaller unexplained part. In this light, the stock market excess volatility seems quantitatively less at odds with rationality of the stock market.

#### 4.7. RISK AND UNCERTAINTY PREMIUMS

Empirically, the predictability effect far dominates the uncertainty premium effect of approximation errors, but risk and uncertainty premiums in asset pricing models are interesting in their own right. Mehra and Prescott (1985) and the literature initiated by this study show that traditional rational expectation models are unable to explain the high empirical premium on stocks. This *equity premium puzzle* is particularly troubling for Lucas (1978)-type models with endogenous price determination, since the price volatility and the resulting premium generated by these models are very low, even compared to Mehra and Prescott (1985)-type models. Empirical rationality and approximation errors are interesting in this context, since a new type of uncertainty and premium are added to the model.

Table 3 shows the risk and uncertainty premiums implied by the four models. All numbers are calculated using the parameter estimates in Panel B of Table 2. In particular, they all share the same relative risk aversion factor,  $\gamma = 1$ . All figures have been calculated for the deterministic steady state values of  $c$  and  $d$ . Moreover, the figures have been calculated as the average of the premiums implied by the observed empirical series of  $p$ ,  $d$ , and  $c$ . Since the effects are not constant over the state space, the empirical premiums vary. The standard deviation is calculated to illustrate the size of this variation.

Starting with the dividend risk, the risk premium lowers the equilibrium price. The fall is approximately 0.5% for all models, but slightly higher for the additive than the multiplicative specification. This difference should come as no surprise, since the additive specification assigns higher probability to the worst case scenarios. The sign of the price change is expected, too. Since the shocks to consumption and dividends are positively correlated, dividends are a poor insurance against future consumption uncertainty.

Turning to the premiums attributed to the existence of approximation errors, these are obviously only available for empirical rationality models. Again, positive correlation between the shocks to consumption and to approximation errors causes the equilibrium price to fall. Although the size of the fall now differs more across the specifications, the effect is roughly of the same magnitude as the traditional risk premium for all specifications. This result is of interest for the equity premium puzzle. Although the model is never close to explaining the equity premium puzzle, the stock premium of the traditional Lucas (1978)-model could very well be doubled due to approximation errors.

Finally, the risk premium of consumption uncertainty is rather constant across models, and raises the price of the stock by 0.3%. A price increase is expected, since consumption uncertainty increases the stock demand for consumption smoothing purposes.

#### 4.8. EXPLOITING EXCESS RETURN OPPORTUNITIES

The descriptive success of  $p^T$  is limited, compared to  $p^E$ . However, due to the strong normative foundation of the original theoretical model, observed price deviations from  $p^T$  might offer excess return opportunities to the rational investor. Assume that the theoretical model in (4) and (5.A) is a precise description of the rational investor's micro-situation, and assume that the investor uses the model for expectation formation (theoretical expectations). Consider the utility gained by giving up consumption worth one unit of utility at time  $t - 1$ , investing the funds in stocks, and consuming the total value of the investment at time  $t$ . Let  $\partial u_t$  denote the overall utility gained by this transaction. According to the Euler equation, the expected utility gain should be zero in equilibrium. However, the realized gain might differ, of course.

Table 4 shows a few statistics on the relation between the actual realized utility gains  $\partial u_t$  and the one-year-in-advance expected utility gains  $\partial u_t^e$  based on the theoretical rationality model  $M_A^T$ . There is significant linear dependence between  $\partial u_t$  and  $\partial u_t^e$ , and the correlation is almost 0.25. The theoretical model is therefore capable of predicting utility gain opportunities, but with a regression coefficient at

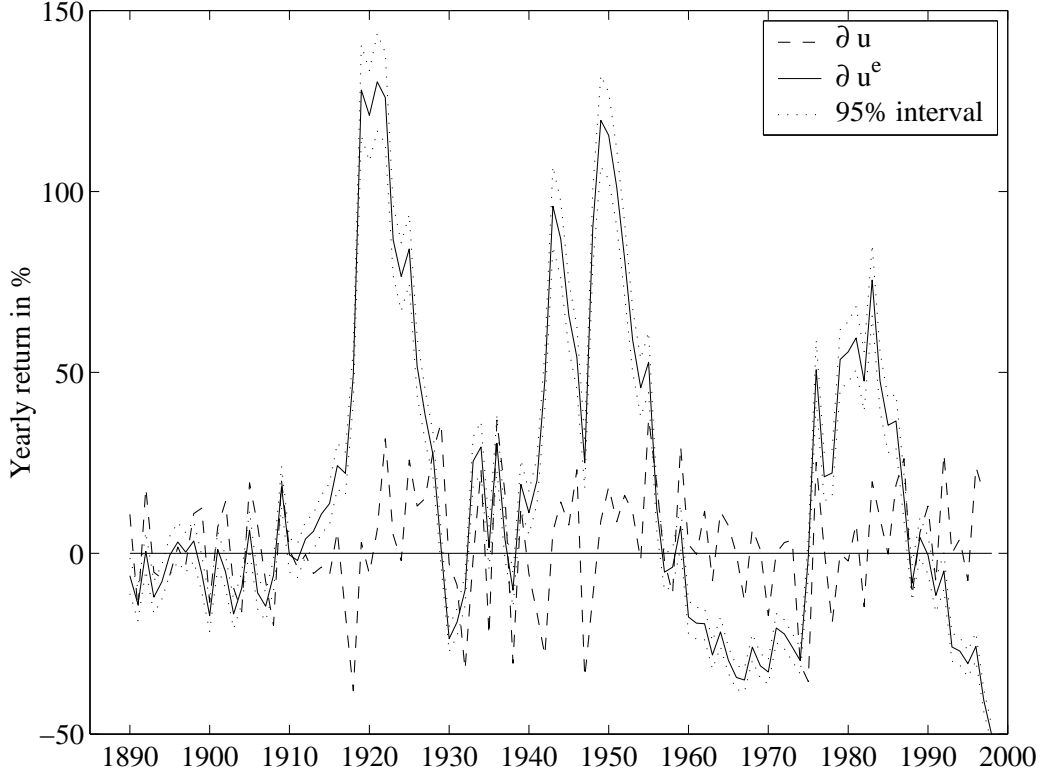


FIGURE 4: Utility gains with theoretical expectations.

Expectations with respect to future return opportunities based at a theoretical model:  $M_A^T$ . The confidence interval is calculated using quasi monte carlo integration, (1000 Niederreider points).

0.1, the model is overestimating the size of utility gain opportunities with a factor 10.

Figure 4 shows  $\partial u_t$  and  $\partial u_t^e$  together with the one-year-in-advance 95% confidence interval based on  $M_A^T$ . Obviously, the confidence with which the theoretical model predicts  $\partial u_t$  is far too optimistic. Only 17% of the realized  $\partial u_t$ -values fall inside the 95% confidence interval. Although the investment strategy implied by the model earns a positive utility gain in the long run, Figure 4 shows that there are at least 3 periods in the sample that would have ruined the investors who took the full consequence of the model's predictions: In the years around 1920, 1950, and 1980 the model would have suggested to borrow, if possible, at a fixed yearly interest rate of more than 50%, and investing the funds in the S&P index. All in all, the investment strategy suggested by theoretical rationality seems to be far too aggressive.<sup>14</sup>

Now consider a similar experiment with empirical rationality imposed. Assume again that the micro-situation of the investors is well described by (4) and (5.A).

<sup>14</sup>It should be noticed that the general equilibrium model does not allow the investor to deviate from  $w = 0$ . Thus, strictly speaking, only marginal changes can be analyzed.

TABLE 4  
EXCESS RETURN STATISTICS

Model	$\partial u_t = \alpha \partial u_t^e + \epsilon_t$					
	$\alpha$	$R^2$	DW	$Std(\partial u_t)$	$Std(\partial u_t^e)$	$Corr$
$M_A^T$	0.100 (0.033)	0.058	1.842	16.33	42.44	0.2436
$M_A^E$	0.914 (0.310)	0.054	1.830	16.33	4.713	0.2426

*Notes:*  $\alpha$  is the least square regression coefficient. DW is the Durbin-Watson test statistic.  $Std(\partial u_t)$  and  $Std(\partial u_t^e)$  are standard deviations for actual and expected utility return. “Corr” is the correlation between  $\partial u_t$  and  $\partial u_t^e$ .

However, this time investors realize that at the macro-level the price will be subject to approximation errors as described by  $M_A^E$ . Therefore, investors form empirical expectations.

The introduction of empirical expectations improves the normative properties significantly. The relevant  $\alpha$ -estimate in Table 4 is no longer significantly different from 1. Thus, the size of  $\partial u_t$  is correctly predicted on average by  $M_A^E$ . Figure 5, which should be compared with Figure 4, shows  $\partial u_t$  together with  $\partial u_t^e$  and the 95% confidence interval based on the empirical rationality model  $M_A^E$ . With 96.3% of the predictions inside the 95% significance band, also the uncertainty of investing is accurately predicted by the empirical model.

Figure 5 might explain why serious mispricing is not eliminated by *fundamentalists*. The expected gain might be positive, but the uncertainty involved in speculating against the market is considerable. Especially, the uncertainty seems to increase in the years around 1920, 1950, and 1980, when investments in the S&P index are predicted to be most profitable. Therefore, investment strategies that recognize approximation errors are more defensive.

## 5. CONCLUSION

This paper has presented an alternative to the traditional working hypothesis that considers structural economic models to be true. This hypothesis causes practical problems to the researcher, in the form of statistical degeneracy. Moreover, rational agents are assumed to ignore empirical information that might be important. These problems were dealt with using an approximation error approach to model uncertainty, combined with a new notion of empirical rationality on the part of the agent. Both the researcher and the agents recognize that the implications of the model are subject to approximation errors. Nevertheless, it was possible to keep the analysis in the normative tradition of Savage (1954).

The importance of the approach was illustrated using a Lucas (1978)-type asset pricing model, modified to accommodate the empirical rationality principle, and

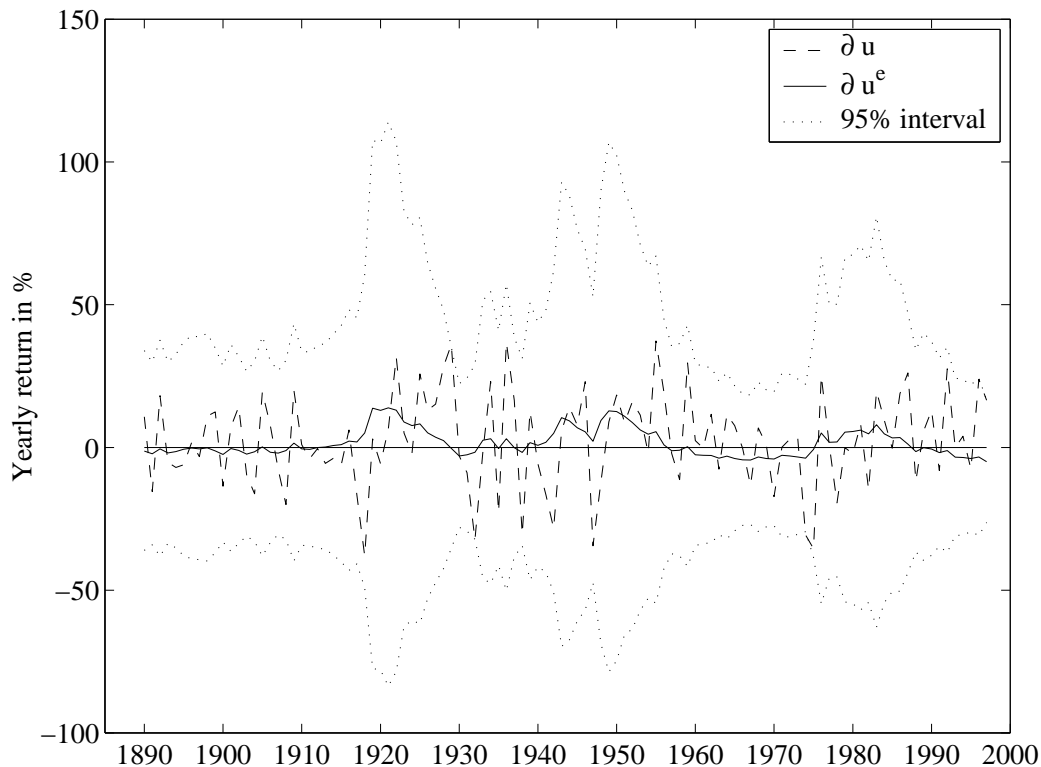


FIGURE 5: Utility gains with empirical expectations

Expectations with respect to future return opportunities based at an empirical model:  $M_A^E$ . The confidence interval is calculated using quasi Monte Carlo integration, (1000 Niederreider points).

applied to U.S. stock prices. The significant difference between the empirical and equilibrium prices was accounted for by introducing approximation errors and allowing the representative investor to re-optimize the investment decision. The new equilibrium price showed significant descriptive as well as normative improvements.

From a descriptive perspective, two important stock market puzzles were addressed. The *stock market excess volatility* documented by Shiller (1981) and Grossman and Shiller (1981) was significantly reduced. Due to the sensitivity of the equilibrium price to observed approximation errors, the distance between the empirical price and the equilibrium price was reduced by 90%.

In addition, the *equity premium puzzle* documented by Mehra and Prescott (1985) was addressed. Approximation errors represent uncertainty about the appropriate asset pricing model, and this uncertainty requires an uncertainty premium. The equity premium puzzle was not solved, but the analysis showed that the total premium was easily doubled due to approximation errors.

Also the normative properties of the model were improved. The original theoretical model predicts excess return opportunities far greater than those actually realized. Moreover, the model significantly underestimates the risk involved in exploiting these opportunities. When approximation errors are taken into account,

the expected gains as well as the uncertainty involved in exploiting them are predicted without bias. Thus, the empirical model provides much better advices for stock market investments.

The normative results suggest that investors should take the empirical shortcomings of asset pricing models into account. The descriptive results suggest that investors in fact do that. If this is true, the researcher may consider approximation errors for two reasons. First, the descriptive and normative relevance of the models improve. Secondly, the symmetry of the researcher's and the agents' beliefs are reestablished.

## APPENDIX A RESULTS FOR THE MULTIPLICATIVE SPECIFICATION

This appendix presents a number of closed form results for the asset pricing model in (4) with the multiplicative specifications in (5.M) and (17).

### A.1 CLOSED FORM SOLUTION OF THE EMPIRICAL MODEL

Consider equation (14) in section 3:

$$(22) \quad p^E(d_0, c_0, z_0) = E_0 \sum_{t=1}^{\infty} \beta^t \frac{u'(c_t)}{u'(c_0)} d_t z_{t-1} z_{t-2} \dots z_1.$$

Since

$$\begin{aligned} d_t &= d_0^{\delta^t} \exp(\epsilon_t^d + \delta \epsilon_{t-1}^d + \delta^2 \epsilon_{t-2}^d + \dots + \delta^{t-1} \epsilon_1^d), \\ c_t &= c_0^{\theta^t} \exp(\epsilon_t^c + \theta \epsilon_{t-1}^c + \theta^2 \epsilon_{t-2}^c + \dots + \theta^{t-1} \epsilon_1^c), \\ z_t &= z_0^{\zeta^t} \exp(\epsilon_t^z + \zeta \epsilon_{t-1}^z + \zeta^2 \epsilon_{t-2}^z + \dots + \zeta^{t-1} \epsilon_1^z), \end{aligned}$$

and

$$\begin{aligned} & z_t z_{t-1} \dots z_1 \\ &= z_0^{\zeta^t + \zeta^{t-1} + \dots + \zeta} \exp(\epsilon_t^z + (1 + \zeta) \epsilon_{t-1}^z + \dots + (1 + \zeta + \dots + \zeta^{t-1}) \epsilon_1^z) \\ &= z_0^{\frac{\zeta - \zeta^{t+1}}{1 - \zeta}} \exp\left(\frac{1 - \zeta}{1 - \zeta} \epsilon_t^z + \frac{1 - \zeta^2}{1 - \zeta} \epsilon_{t-1}^z + \dots + \frac{1 - \zeta^t}{1 - \zeta} \epsilon_1^z\right), \end{aligned}$$

equation (22) can be rewritten as

$$\begin{aligned} & p^E(d_0, c_0, z_0) \\ &= \sum_{t=1}^{\infty} \beta^t c_0^{-\gamma(\theta^t - 1)} d_0^{\delta^t} z_0^{\frac{\zeta - \zeta^t}{1 - \zeta}} E_0 \left[ \exp\left( \epsilon_t^d + \delta \epsilon_{t-1}^d + \dots + \delta^{t-1} \epsilon_1^d \right. \right. \\ & \quad \left. \left. - \gamma(\epsilon_t^c + \theta \epsilon_{t-1}^c + \dots + \theta^{t-1} \epsilon_1^c) \right. \right. \\ & \quad \left. \left. + \frac{1 - \zeta^0}{1 - \zeta} \epsilon_t^z + \frac{1 - \zeta^1}{1 - \zeta} \epsilon_{t-1}^z + \dots + \frac{1 - \zeta^{t-1}}{1 - \zeta} \epsilon_1^z \right) \right] \\ &= \sum_{t=1}^{\infty} \beta^t c_0^{-\gamma(\theta^t - 1)} d_0^{\delta^t} z_0^{\frac{\zeta - \zeta^t}{1 - \zeta}} E_0 \exp\left( f_0^\top \epsilon_t + f_1^\top \epsilon_{t-1} + \dots + f_{t-1}^\top \epsilon_1 \right), \end{aligned}$$



where

$$f_s = \begin{bmatrix} \delta^s \\ -\gamma\theta^s \\ \frac{1-\zeta^s}{1-\zeta} \end{bmatrix} \quad \text{and} \quad \epsilon_s = \begin{bmatrix} \epsilon_s^d \\ \epsilon_s^c \\ \epsilon_s^z \end{bmatrix}.$$

Calculating the expected value of the lognormal distributed variables gives the equilibrium price,

$$\begin{aligned} p^E(d_0, c_0, z_0) &= \sum_{t=1}^{\infty} \beta^t c_0^{-\gamma(\theta^t-1)} d_0^{\delta^t} z_0^{\frac{\zeta-\zeta^t}{1-\zeta}} \times \exp\left(\mu^\top f_0 + \frac{1}{2} f_0^\top \Omega f_0\right. \\ &\quad \left. + \mu^\top f_1 + \frac{1}{2} f_1^\top \Omega f_1\right. \\ &\quad \left. \vdots\right. \\ &\quad \left. + \mu^\top f_{t-1} + \frac{1}{2} f_{t-1}^\top \Omega f_{t-1}\right) \\ &= \sum_{t=1}^{\infty} \beta^t c_0^{-\gamma(\theta^t-1)} d_0^{\delta^t} z_0^{\frac{\zeta-\zeta^t}{1-\zeta}} \Psi_t, \end{aligned}$$

where

$$\Psi_t = \exp\left(\sum_{s=0}^{t-1} h_s\right) \quad \text{and} \quad h_s = \mu^\top f_s + \frac{1}{2} f_s^\top \Omega f_s.$$

## A.2 CLOSED FORM SOLUTION OF THE THEORETICAL MODEL

The theoretical model is a special case of the empirical model where  $z_t \equiv 1$  for all  $t$ :

$$p^E(d_0, c_0, z_0) = \sum_{t=1}^{\infty} \beta^t c_0^{-\gamma(\theta^t-1)} d_0^{\delta^t} \Lambda_t,$$

where  $\Lambda_t$ , is defined like  $\Psi_t$  except that

$$f_s^T = \begin{bmatrix} \delta^s \\ -\gamma\theta^s \\ 0 \end{bmatrix}$$

replaces  $f_s$ .

## A.3 DETERMINATION OF $\mu$

Consider first the dividend process. For given  $\delta$  and  $\sigma_d^2$ ,  $\mu_d$  is determined such that the unconditional expectation of the dividend is one:

$$E[d_t] = 1.$$

Using the transition equation for dividends and repeated substitution gives

$$\begin{aligned}
d_t &= d_{t-1}^\delta \exp \epsilon_t^d \\
&= (d_{t-2}^\delta \exp \epsilon_{t-1}^d)^\delta \exp \epsilon_t^d \\
&= d_{t-2}^{\delta^2} \exp(\epsilon_t^d + \delta \epsilon_{t-1}^d) \\
&= d_{t-\tau}^{\delta^\tau} \exp(\epsilon_t^d + \delta \epsilon_{t-1}^d + \delta^2 \epsilon_{t-2}^d + \dots + \delta^{\tau-1} \epsilon_{t-\tau+1}^d) \\
&\rightarrow \exp(\epsilon_t^d + \delta \epsilon_{t-1}^d + \delta^2 \epsilon_{t-2}^d + \dots) \text{ as } \tau \rightarrow \infty.
\end{aligned}$$

Hence,

$$\begin{aligned}
E[d_t] &= \exp \left( (1 + \delta + \delta^2 + \dots) \mu_d + \frac{1}{2} (1 + \delta^2 + \delta^4 + \dots) \sigma_d^2 \right) \\
&= \exp \left( \frac{1}{1-\delta} \mu_d + \frac{1}{2(1-\delta^2)} \sigma_d^2 \right),
\end{aligned}$$

and

$$\mu_d = -\frac{1-\delta}{2(1-\delta^2)} \sigma_d^2 \Rightarrow E[d_t] = 1.$$

Using the same arguments for the consumption process gives

$$\mu_c = -\frac{1-\theta}{2(1-\theta^2)} \sigma_c^2 \Rightarrow E[c_t] = 1.$$

To determine  $\mu_z$  it should be noted that  $f_i^\top \rightarrow [0 \ 0 \ \frac{1}{1-\zeta}]$  as  $i \rightarrow \infty$ . This causes

$$h_s \rightarrow \frac{1}{1-\zeta} \mu_z + \frac{1}{2(1-\zeta)^2} \sigma_z^2 \quad \text{for } s \rightarrow \infty.$$

Therefore,

$$\Psi_t = \exp \left( \sum_{s=0}^{t-1} h_s \right)$$

increases exponentially if the limit value of  $h_s$  is positive. Hence, the equilibrium price is infinite. To ensure a finite equilibrium price,  $\mu_z$  must be bounded by

$$\mu_z \leq \frac{-\sigma_z^2}{2(1-\zeta)}.$$

Thus,  $\mu_z = \frac{-\sigma_z^2}{2(1-\zeta)}$  is imposed in the empirical analysis.

#### A.4 DEFINITION OF RISK PREMIUMS AND THEIR CLOSED FORM EXPRESSIONS

Using the following asset pricing functions,

$$\begin{aligned}
p_0^E(\bar{z}) &= E_0 \left[ \sum_{t=1}^{\infty} \tilde{\beta}^t d_t E_0(z_{t-1} z_{t-2} \cdots z_1) \right], \\
p_0^E(\bar{d}) &= E_0 \left[ \sum_{t=1}^{\infty} \tilde{\beta}^t E_0(d_t) z_{t-1} z_{t-2} \cdots z_1 \right], \\
(23) \quad p_0^E(\bar{d}, \bar{z}) &= E_0 \left[ \sum_{t=1}^{\infty} \tilde{\beta}^t E_0(d_t) E_0(z_{t-1} z_{t-2} \cdots z_1) \right], \\
p_0^E(\bar{d}, \bar{c}, \bar{z}) &= E_0 \left[ \sum_{t=1}^{\infty} \beta^t \frac{u'(E_0(c_t))}{u'(c_0)} E_0(d_t) E_0(z_{t-1} z_{t-2} \cdots z_1) \right],
\end{aligned}$$

the risk premiums with respect to  $z$ ,  $d$ , and  $c$  are defined as

$$\begin{aligned}
\pi_0^z &= E_0 \left[ \sum_{t=1}^{\infty} \tilde{\beta}^t E_0(d_t) (z_{t-1} z_{t-2} \cdots z_1 - E_0(z_{t-1} z_{t-2} \cdots z_1)) \right] \\
&= p_0^E(\bar{d}) - p_0^E(\bar{d}, \bar{z}), \\
\pi_0^d &= E_0 \left[ \sum_{t=1}^{\infty} \tilde{\beta}^t (d_t - E_0(d_t)) E_0(z_{t-1} z_{t-2} \cdots z_1) \right] \\
&= p_0^E(\bar{z}) - p_0^E(\bar{d}, \bar{z}), \\
\pi_0^c &= E_0 \left[ \sum_{t=1}^{\infty} \beta^t \frac{u'(c_t) - u'(E_0(c_t))}{u'(c_0)} E_0(d_t) E_0(z_{t-1} z_{t-2} \cdots z_1) \right] \\
&= p_0^E(\bar{d}, \bar{z}) - p_0^E(\bar{d}, \bar{c}, \bar{z}).
\end{aligned}$$

Since all the pricing functions in (23) are available in closed form, this is also the case for the risk premiums. Consider first  $p_0^E(\bar{z})$ :

$$\begin{aligned}
p_0^E(\bar{z}) &= \sum_{t=1}^{\infty} \beta^t c_0^{-\gamma(\theta^t-1)} d_0^{\delta^t} z_0^{\frac{\zeta-\zeta^t}{1-\zeta}} \\
&\quad \times E_0 \left[ \exp \left( \epsilon_t^d + \delta \epsilon_{t-1}^d + \cdots + \delta^{t-1} \epsilon_1^d - \gamma(\epsilon_t^c + \theta \epsilon_{t-1}^c + \cdots + \theta^{t-1} \epsilon_1^c) \right) \right] \\
&\quad \times E_0 \left[ \exp \left( \frac{1-\zeta^0}{1-\zeta} \epsilon_t^z + \frac{1-\zeta^1}{1-\zeta} \epsilon_{t-1}^z + \cdots + \frac{1-\zeta^{t-1}}{1-\zeta^0} \epsilon_1^z \right) \right] \\
&= \sum_{t=1}^{\infty} \beta^t c_0^{-\gamma(\theta^t-1)} d_0^{\delta^t} z_0^{\frac{\zeta-\zeta^t}{1-\zeta}} \exp \left( \sum_{s=0}^{t-1} \mu^\top f_s + \frac{1}{2} f_s^\top \Omega^z f_s \right)
\end{aligned}$$

which is equal to the closed form solution to  $p_0^E$  except that

$$\Omega^z = \begin{bmatrix} \sigma_d^2 & \sigma_{d,c} & 0 \\ \sigma_{d,c} & \sigma_c^2 & 0 \\ 0 & 0 & \sigma_z^2 \end{bmatrix}$$

replaces  $\Omega$ . Likewise, the closed form solutions for  $p_0^E(\bar{d})$  and  $p_0^E(\bar{d}, \bar{z})$  equal the solution to  $p_0^E$ , except that  $\Omega$  is replaced by

$$\Omega^{\bar{d}} = \begin{bmatrix} \sigma_d^2 & 0 & 0 \\ 0 & \sigma_c^2 & \sigma_{d,c} \\ 0 & \sigma_{d,c} & \sigma_z^2 \end{bmatrix} \quad \text{and} \quad \Omega^{\bar{d}, \bar{z}} = \begin{bmatrix} \sigma_d^2 & 0 & 0 \\ 0 & \sigma_c^2 & 0 \\ 0 & 0 & \sigma_z^2 \end{bmatrix}.$$

The case of  $p_0^E(\bar{d}, \bar{c}, \bar{z})$  is strictly more complicated due to the non-linearity of the utility function:

$$\begin{aligned} p_0^E(\bar{z}, \bar{c}, \bar{d}) &= \sum_{t=1}^{\infty} \beta^t c_0^{-\gamma(\theta^t-1)} d_0^{\delta^t} z_0^{\frac{\zeta-\zeta^t}{1-\zeta}} \\ &\quad \times E_0 \exp(\epsilon_t^d + \delta \epsilon_{t-1}^d + \dots + \delta^{t-1} \epsilon_1^d) \\ &\quad \times E_0 \exp\left(\frac{1-\zeta^0}{1-\zeta} \epsilon_t^z + \frac{1-\zeta^1}{1-\zeta} \epsilon_{t-1}^z + \dots + \frac{1-\zeta^{t-1}}{1-\zeta^0} \epsilon_1^z\right) \\ &\quad \times \left(E_0 \exp(\epsilon_t^c + \theta \epsilon_{t-1}^c + \dots + \theta^{t-1} \epsilon_1^c)\right)^{-\gamma}. \end{aligned}$$

Since

$$\begin{aligned} &\left(E_0 \exp(\epsilon_t^c + \theta \epsilon_{t-1}^c + \dots + \theta^{t-1} \epsilon_1^c)\right)^{-\gamma} \\ &= \exp\left(-\gamma \mu_c - \gamma \frac{1}{2} \sigma_c^2 - \gamma \theta \mu_c - \gamma \frac{1}{2} \theta^2 \sigma_c^2 - \dots - \gamma \theta^{t-1} \mu_c - \gamma \frac{1}{2} \theta^{2(t-1)} \sigma_c^2\right) \end{aligned}$$

the closed form solution for  $p_0^E(\bar{d}, \bar{c}, \bar{z})$  equals the solution for  $p_0^E$ , except that

$$\Omega^{\bar{d}, \bar{z}, \bar{c}} = \begin{bmatrix} \sigma_d^2 & 0 & 0 \\ 0 & -\frac{\sigma_c^2}{\gamma} & 0 \\ 0 & 0 & \sigma_z^2 \end{bmatrix}$$

replaces  $\Omega$ .

## APPENDIX B NUMERICAL SOLUTION METHOD

The numerical solutions of the models with additive specifications are projection method solutions to the Euler-equations, see Judd (1992).

The approximation basis consists of tensor cubic b-splines, see de Boor (1978). For the theoretical and the empirical model,  $5 \times 5$  and  $5 \times 5 \times 5$  b-spline elements were used. Integration with respect to expectations were calculated using  $4 \times 4$  and  $2 \times 2 \times 2$  Hermite points. The size of the state space was chosen such that all integration points were interior. The b-spline coefficients were chosen to minimize the squared numerical approximation errors at  $13 \times 13$  and  $13 \times 13 \times 13$  approximation points equally spaced over the state space.

With respect to precision, the numerical solution seemed to violate the Euler equation with a maximum of 1E-4% over the chosen state space. This precision is, of course, conditioned on the precision of the integration. However, the quasi Monte Carlo methods used for calculating the confidence intervals in figures 4 and 5 also showed that the Hermite integration was quite accurate despite the small number of points.

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