WP 2001-2

Mean variance efficient portfolios by linear programming: A review of some portfolio selection criteria of Elton, Gruber and Padberg

af

Bjarne Astrup Jensen

INSTITUT FOR FINANSIERING, Handelshøjskolen i København
Solbjerg Plads 3, 2000 Frederiksberg C
tlf.: 38 15 36 15 fax: 38 15 36 00

DEPARTMENT OF FINANCE, Copenhagen Business School
Solbjerg Plads 3, DK - 2000 Frederiksberg C, Denmark
Phone (+45)38153615, Fax (+45)38153600
www.cbs.dk/departments/finance

ISBN 87-90705-47-5
ISSN 0903-0352

Mean variance efficient portfolios by linear programming: A review of some portfolio selection criteria of Elton, Gruber and Padberg

Bjarne Astrup Jensen*
Department of Finance
Copenhagen Business School
Solbjerg Plads, A5.04
DK-2000 Frb., Denmark
Phone: +45 3815 3614 Fax: +45 3815 3600
e-mail:fibj@cbs.dk

JEL Classification: G11

This version:
2nd February 2001

*I thank Chris Blake, Anders Damgaard, Martin Gruber and participants at the French Finance Association's Annual Meeting in Aix-en-Provence for helpful comments on earlier versions of the paper. Part of this research was carried out while the author enjoyed the hospitality of CEREG, Université de Paris IX, Dauphine, as "chercheur invité". Financial support from the Danish Social Science Research Council is gratefully acknowledged.

Abstract: Finding the mean-variance efficient frontier is a quadratic programming problem with an analytical solution, whenever the portfolio choice is unrestricted. The analytical solution involves an inversion of the covariance matrix. When short-sale constraints are added to the problem it is usually thought of as adding considerable complexity to the quadratic programming problem. This paper shows that such problems can be handled by a simple linear programming procedure, which allows for multiple changes of basis variables. We show how some classical selection criteria from models with particular covariance matrices fall into this framework. Furthermore, adding linear constraints like maximum placement limits for subsets of assets is easily incorporated.

Keywords: Mean variance efficient portfolios, short sale constraints, linear programming, multiple basis shifts, placement limits.

1 Introduction

Finding the mean-variance efficient frontier is a quadratic programming problem with an analytical solution, whenever the portfolio choice is unrestricted. The analytical solution involves an inversion of the covariance matrix. When short-sale constraints are added to the problem it is usually thought of as adding considerable complexity to the quadratic programming problem.

The purpose of this paper is to show that the problem of finding the mean-variance efficient frontier with short-sale constraints can be solved as a linear programming problem. Furthermore, due to the specific structure of the problem, it allows for multiple basis changes during the course of running the simplex algorithm. Although powerful quadratic programming algorithms – and, similarly, general constrained optimization algorithms – exist for solving such problems they usually do not exploit any special structure of the portfolio problem. This paper provides an algorithm that is straightforward to implement by anyone with a basic knowledge of the simplex algorithm and computer programming.

The motivation behind this study derives from a rereading of a number of papers from the late 1970'ties by Elton, Gruber and Padberg,[1] where computationally simple routines for finding mean-variance efficient portfolios are outlined for a number of special cases. These routines were particularly well suited to solve the case with short sale constraints that was otherwise perceived as being computationally burdensome, and for the simpler cases their calculations were reduced to a "back of an envelope" level of complexity. These routines have since then been an integrated part of the widely used textbook by Elton and Gruber.[2]

The exposition – in the original papers as well as in the textbook – is based on elaborately writing out the first order conditions as linear equations and then trying to show the correctness of a postulated solution. While this works relatively smooth in the "one-dimensional" cases, called the "single index model" and the "constant correlation model", the exact implementation is not spelled out explicitly for any of the "multi-dimensional" cases discussed under headlines such as "multi-index models" or "multi-group models". In Elton, Gruber, and Padberg (1977), e.g., it is stated (p. 336) that ".. The following seems to us to be an efficient method. ..", but no convergence proofs actually exist.[3]

In this paper we show that the problem of finding the tangency portfolio under short sale constraints, which was the original problem to which computationally simple solutions were sought, fits into the framework of *Wolfe's quadratic simplex algorithm* as a rather simple example. That is, a convergent LP-algorithm with computationally simple steps always exists as one way of solving the problem. The simplicity of the LP-solution is enhanced by the fact that the structure of the problem allows for multiple changes of basis variables in each step.

For the particular cases discussed in the above-mentioned papers we show that the simplicity of the solutions originally proposed derive from the same simple matrix inversion lemma. Additionally we present an alternative proof, based on linear programming, of the optimality of the "cut-off procedure" originally described in the "one-dimensional" cases. The same methodology only applies to the "multi-dimensional" cases under a certain reinterpretation, but the LP-algorithm keeps track of included and excluded assets in a systematic way under all circumstances.

[1] The references in question are Elton, Gruber, and Padberg (1976), Elton, Gruber, and Padberg (1977), Elton, Gruber, and Padberg (1978a), Elton, Gruber, and Padberg (1978b), Elton, Gruber, and Padberg (1979).

[2] See Elton and Gruber (1995), chapters 7-9. The computational routines are outlined in some detail in an appendix to chapter 9.

[3] Gruber (1997).

An additional application of the LP-algorithm is to trace out the entire efficient frontier with short-sale constraints by varying the value of the risk-free rate of interest as a parameter. Performing sensitivity analysis on the solution provides the answer to the composition of the efficient portfolios as well as the location of the critical points along the frontier, where the set of included assets changes. That is, the entire efficient frontier can be traced out by solving *one* linear programming problem and subsequently perform a standard sensitivity analysis.

We also demonstrate that adding a number of additional linear restrictions to the problem does not destroy the simplicity of the LP-algorithm. Such linear restrictions are typically portfolio allocation limits like "maximum 40% of assets within a specific group" or "no individual asset may make up more than 20% of the entire portfolio value". It is important, however, that the mean-variance efficient set is continuous and concave, which is the case when such additional constraints are linear. Problems with cardinality constraints or constraints of a binary character, as described in e.g. Beasley et al. (2000), may lead to discontinuties and/or non-cancavity of the set of feasible portfolio allocations. Such problems cannot be handled by the LP-algorithm developed in this paper.

The paper is organised as follows.

In section 2 we introduce the notation and the well-known mathematical programming problem of finding the tangency portfolio. It is shown how the problem of finding the tangency portfolio can be formulated as a linear programming problem with a certain restriction attached to it. In the Appendix we provide a separate proof of convergence of this LP-algorithm.

In section 3 the matrix inversion lemma is stated and applied to the special covariance matrices employed.

In section 4 the problem of computing the efficient frontier, when returns are described by the "single index model", is formulated as a linear programming problem and as the premier example of a one-dimensional model. It is shown how the procedure developed by Elton, Gruber, and Padberg (1976) for solving this problem, relying on a ranking by the Treynor ratios, can be derived from a pivoting scheme which guarantees that once an asset is included in the basis it will never leave the basis again.

In section 5 it is shown that the so-called "constant correlation model" has a reduced form that is equivalent to one particular example of the single index model. Additionally, a simple numerical example with 3 assets satisfying the constant correlation model is presented.

In section 6 the "multi-group model" is put into the linear programming framework. It is shown how the pivoting scheme also for this multi-dimensional model is in accordance with the originally developed ranking device. However, the linear programming routine enables the establishment of an order in which to change assets in the portfolio that is computationally simple and guarantees convergence. The numerical example in Elton, Gruber, and Padberg (1977) is used for illustration.

In section 7 the "multi-index model" is outlined. Like any model with any covariance matrix the multi-index model fits into the linear programming framework, but in its *general* form the multi-index model does not give rise to a ranking procedure that generates an algorithmic shortcut. However, reinterpreting the model by assuming that a market portfolio with no residual risk exists the model becomes mathematically equivalent to the multi-group model. Details of the solution are spelled out in the Appendix.

Finally, in section 8 we show that the LP-algorithm can easily be generalised to handle linear constraints like e.g. maximum placement limits.

2 The portfolio selection problem

In the standard mean-variance portfolio selection model for N risky assets, the model input in the presence of a riskless investment opportunity is

- the vector $\overline{\boldsymbol{R}} \in \mathbb{R}^N$ of expected returns
- the covariance matrix $\boldsymbol{\Omega} \in \mathbb{R}^N \times \mathbb{R}^N$
- a riskless rate of interest R_f

In addition to this notation we will use the symbol $\mathbf{1}$ for the vector of one's in $\mathbb{R}^N$, i.e.

$$\mathbf{1}^{\boldsymbol{t}} = (1, 1, \ldots, 1) \tag{2.1}$$

It is well known that without short sale constraints any mean-variance efficient portfolio is a portfolio of two assets:[4] The riskless asset and one particular portfolio – the *tangency portfolio* – composed solely of risky assets. The composition of the tangency portfolio is derived in a straightforward manner from the relevant first order condition:

$$\boldsymbol{\Omega Z} = \overline{\boldsymbol{R}} - R_f \mathbf{1} \tag{2.2}$$

by normalizing the vector $\boldsymbol{Z}$ to a portfolio, i.e. such that the sum of its components add up to one. This calculation is also useful when there is no riskless asset, since R_f can be treated as a free parameter. By varying R_f the entire efficient frontier for the case with no riskless asset can be traced out.

In the case with short sale constraints the relevant first order conditions or Kuhn-Tucker conditions are:

$$\begin{aligned} \boldsymbol{\Omega Z} - \boldsymbol{M} &= \overline{\boldsymbol{R}} - R_f \mathbf{1} && (2.3) \\ Z_i, M_i &\geq 0 \quad i = 1, 2, \cdots, N && (2.4) \\ Z_i M_i &= 0 \quad i = 1, 2, \cdots, N && (2.5) \end{aligned}$$

and the optimal portfolio is found by normalizing the vector $\boldsymbol{Z}$. Again, any mean-variance efficient portfolio is a portfolio of the riskless asset and one particular portfolio composed solely of risky assets. The problem is that determining the *composition* of this portfolio involves determining which assets to include and which assets to exclude due to short sale constraints. As in the case with no constraints this calculation is also one way of tracing out the entire mean-variance efficient frontier when there is no riskless asset.

The Lagrangian multipliers M_i can be interpreted as the additional risk premium necessary in order for assets excluded by the short-sale constraint to be marginal investments with portfolio weight $Z_i = 0$ in an optimal unconstrained solution.

The solution in both cases involves an inversion of $\boldsymbol{\Omega}$ or the appropriate subset of $\boldsymbol{\Omega}$, either directly or indirectly through solving a set of linear equations. The simplified portfolio selection procedures developed by Elton, Gruber and Padberg reduces the computational burden by postulating special structures on $\boldsymbol{\Omega}$.

[4] Besides the sources already refererred to by Elton, Gruber and Padberg, see Huang and Litzenberger (1988), chapter 3, Merton (1972) or Roll (1977).

The case with short sale constraints turns out to be a linear programming problem. Hence, it is computationally much easier than the general quadratic programming problem solved by one of the available *general* methods for such problems. A solution to the Kuhn-Tucker conditions (2.3)-(2.5) can be obtained as an optimal solution to the following problem:

$$\begin{gathered} \text{Min} \sum_{p=1}^{N} X_p \\ \text{subject to} \\ \boldsymbol{\Omega Z} - \boldsymbol{M} + \boldsymbol{X} = \overline{\boldsymbol{R}} - R_f \mathbf{1} \\ \boldsymbol{Z},\ \boldsymbol{M},\ \boldsymbol{X} \geq \mathbf{0} \\ \boldsymbol{Z} \cdot \boldsymbol{M} = \mathbf{0} \end{gathered} \tag{2.6}$$

This mathematical programming problem differs from a linear programming problem only by the so-called *exclusion rule* $\boldsymbol{Z} \cdot \boldsymbol{M} = 0$. I.e. if Z_i is in the basis, M_i must not enter the basis. And vice versa.

It is known that by performing usual simplex iterations, according to what is known as *Wolfe's quadratic simplex algorithm*, in order to decrease the value of the objective function to zero, the enforcement of the exclusion rule as a restriction on the choice of the in-coming basis variable does not prevent the LP-routine from reaching an optimal solution[5]. This is proven separately in the Appendix, since for this particular case Wolfe's algorithm is notationally much simpler than the general case and can be improved by allowing for multiple changes of basis.

Throughout, the set of indices corresponding to basis X-variables are identified by the binary N-vector $\mathbf{1}_{\boldsymbol{x}}$. The set of non-basis X-variables are identified by the complimentary binary N-vector $\mathbf{1}_{\boldsymbol{n_x}}$.

One initial and primal feasible basis solution is obvious: Select *either* X_i *or* M_i according to whether $\overline{R}_i - R_f$ is positive or negative and set $\boldsymbol{Z} \equiv \mathbf{0}$. With this choice of basis variables the initial simplex tableau looks as follows:

X_0	$\boldsymbol{Z}$	$\boldsymbol{M}$	$\boldsymbol{X}$	rhs
1	$\mathbf{1}_{\boldsymbol{x}}^{\boldsymbol{t}}\boldsymbol{\Omega}$	$-\mathbf{1}_{\boldsymbol{x}}^{\boldsymbol{t}}$	$-\mathbf{1}_{\boldsymbol{n_x}}^{\boldsymbol{t}}$	$\mathbf{1}_{\boldsymbol{x}}^{\boldsymbol{t}}\left(\overline{\boldsymbol{R}} - R_f\mathbf{1}\right)$
$\mathbf{0}$	$\boldsymbol{\Omega}$	$-\boldsymbol{I}$	$\boldsymbol{I}$	$\overline{\boldsymbol{R}} - R_f\mathbf{1}$

where the variable X_0 is added in the conventional manner in order to represent the value of the objective function.

When the algorithm stops the simplex tableau – after a suitable permutation of the variables – looks like:

X_0	$\boldsymbol{Z_b}$	$\boldsymbol{M_b}$	$\boldsymbol{Z_n}$	$\boldsymbol{M_n}$	$\boldsymbol{X}$	rhs
1	$\mathbf{0}^{\boldsymbol{t}}$	$\mathbf{0}^{\boldsymbol{t}}$	$\mathbf{0}^{\boldsymbol{t}}$	$\mathbf{0}^{\boldsymbol{t}}$	$-\mathbf{1}^{\boldsymbol{t}}$	0
0	$\boldsymbol{\Omega_{bb}}$ $\boldsymbol{\Omega_{nb}}$	$\mathbf{0}$ $-\boldsymbol{I}$	$\boldsymbol{\Omega_{bn}}$ $\boldsymbol{\Omega_{nn}}$	$-\boldsymbol{I}$ $\mathbf{0}$	$\boldsymbol{I}\ \mathbf{0}$ $\mathbf{0}\ \boldsymbol{I}$	$\overline{\boldsymbol{R}} - R_f\mathbf{1}$

[5]The *general* structure of the algorithm is well described in e.g. Franklin (1980), pp. 177-187.

with b for basis-variables and n for non-basis variables. In inverted terms[6] this can be written as:

X_0	$\boldsymbol{Z_b}$	$\boldsymbol{M_b}$	$\boldsymbol{Z_n}$	$\boldsymbol{M_n}$	$\boldsymbol{X}$
1	$\mathbf{0}^t$	$\mathbf{0}^t$	$\mathbf{0}^t$	$\mathbf{0}^t$	$-\mathbf{1}^t$
$\mathbf{0}$	$\boldsymbol{I}$	$\mathbf{0}$	$\boldsymbol{\Omega}_{bb}^{-1}\boldsymbol{\Omega}_{bn}$	$-\boldsymbol{\Omega}_{bb}^{-1}$	$\boldsymbol{\Omega}_{bb}^{-1} \quad \mathbf{0}$
	$\mathbf{0}$	$-\boldsymbol{I}$	$\boldsymbol{\Omega}_{nn}-\boldsymbol{\Omega}_{nb}\boldsymbol{\Omega}_{bb}^{-1}\boldsymbol{\Omega}_{bn}$	$\boldsymbol{\Omega}_{nb}\boldsymbol{\Omega}_{bb}^{-1}$	$-\boldsymbol{\Omega}_{nb}\boldsymbol{\Omega}_{bb}^{-1} \quad \boldsymbol{I}$

rhs
0
$\begin{pmatrix} \boldsymbol{\Omega}_{bb}^{-1} & \mathbf{0} \\ -\boldsymbol{\Omega}_{nb}\boldsymbol{\Omega}_{bb}^{-1} & \boldsymbol{I} \end{pmatrix} \left(\overline{\boldsymbol{R}} - R_f\mathbf{1}\right)$

At any intermediate step, the simplex tableau in original terms – after a suitable permutation of the variables – looks like:

X_0	$\boldsymbol{Z_b}$	$\boldsymbol{Z_n}$	$\boldsymbol{M}$	$\boldsymbol{X}$	rhs
1	$\mathbf{0}^t$	$\mathbf{0}^t$	$\mathbf{0}^t$	$-\mathbf{1}^t$	0
$\mathbf{0}$	$\boldsymbol{\Omega}_{bb}$	$\boldsymbol{\Omega}_{bn}$	$-\boldsymbol{I} \quad \mathbf{0}$	$\boldsymbol{I} \quad \mathbf{0}$	$\overline{\boldsymbol{R}} - R_f\mathbf{1}$
	$\boldsymbol{\Omega}_{nb}$	$\boldsymbol{\Omega}_{nn}$	$\mathbf{0} \quad -\boldsymbol{I}$	$\mathbf{0} \quad \boldsymbol{I}$	

In inverted terms – and with a slight abuse of the notation – this tableau becomes[7]:

X_0	$\boldsymbol{Z_b}$	$\boldsymbol{Z_n}$	$\boldsymbol{M}$
1	$\mathbf{0}^t$	$\mathbf{1}_x^t\left[\boldsymbol{\Omega}_{nn}-\boldsymbol{\Omega}_{nb}\boldsymbol{\Omega}_{bb}^{-1}\boldsymbol{\Omega}_{bn}\right]$	$\mathbf{1}_x^t\boldsymbol{\Omega}_{nb}\boldsymbol{\Omega}_{bb}^{-1} \quad -\mathbf{1}_x^t$
$\mathbf{0}$	$\boldsymbol{I}$	$\boldsymbol{\Omega}_{bb}^{-1}\boldsymbol{\Omega}_{bn}$	$-\boldsymbol{\Omega}_{bb}^{-1} \quad \mathbf{0}$
	$\mathbf{0}$	$\boldsymbol{\Omega}_{nn}-\boldsymbol{\Omega}_{nb}\boldsymbol{\Omega}_{bb}^{-1}\boldsymbol{\Omega}_{bn}$	$\boldsymbol{\Omega}_{nb}\boldsymbol{\Omega}_{bb}^{-1} \quad -\boldsymbol{I}$

[6] The lower part of this matrix representation deviates from the standard simplex tableau. The rows are multiplied by -1 for reasons to become clear below.

[7] Again, the rows corresponding to non-basis variables X or, equivalently, the rows corresponding to basis variables M, are multiplied by -1 relative to the standard simplex tableau.

$\boldsymbol{X}$		rhs
$-\mathbf{1}^t - \mathbf{1}_x^t \boldsymbol{\Omega}_{nb}\boldsymbol{\Omega}_{bb}^{-1}$	$-\mathbf{1}_{n_x}^t$	$\mathbf{1}_x^t \left[\overline{\boldsymbol{R}}_x - R_f \mathbf{1}_x - \boldsymbol{\Omega}_{nb}\boldsymbol{\Omega}_{bb}^{-1}\left(\overline{\boldsymbol{R}}_b - R_f \mathbf{1}_b\right)\right]$
$\boldsymbol{\Omega}_{bb}^{-1}$	$\mathbf{0}$	$\begin{pmatrix} \boldsymbol{\Omega}_{bb}^{-1} & \mathbf{0} \\ -\boldsymbol{\Omega}_{nb}\boldsymbol{\Omega}_{bb}^{-1} & \boldsymbol{I} \end{pmatrix} \left(\overline{\boldsymbol{R}} - R_f \mathbf{1}\right)$
$-\boldsymbol{\Omega}_{nb}\boldsymbol{\Omega}_{bb}^{-1}$	$\boldsymbol{I}$	

Observe that there is a one-to-one correspondance between basis (non-basis) variables X_i and the corresponding non-basis (basis) variables M_i *within* the set of indices relating to non-basis variables Z_n.

The usual simplex algorithm with the exclusion rule added will look for a variable to enter, in this case among Z_n or M_n, with the prime purpose of having a variable among X_b leave the basis and simultaneously maintaining primal feasibility. I.e. all basis variables must be kept non-negative.

Observe that once an "artificial" variable X_j has left the basis, it can be omitted from further consideration. This means that if the initial basis is chosen as described above, with M_j or X_j included depending upon the sign of $\overline{R}_i - R_f$, then the algorithm only needs to include X-variables for assets with a positive risk premium $\overline{R}_i - R_f$.

This can be implemented in the following manner, allowing for multiple changes of basis variables.

Theorem 1 *The following algorithm converges to the optimal solution of the modified linear programming problem in (2.6):*

1. *Choose the initial basis as a combination of $M-$ and $X-$variables in accordance with the sign of the risk-premium.*

2. *Select a current non-basis variable Z_i or M_i to enter as a new basis variable. The usual criteria can be applied, i.e. looking for positive reduced cost coefficients, although the exclusion rule must be obeyed.*

3. *Find the pivot element. If the entering variable is Z_i, check whether the pivot element refers to another Z_b-variable. If so, perform the usual pivot operations. If not – and with reference to the permutation above this is revealed the first time the usual pivot ratio in any row below the Z_b rows is encountered to be lower than among the Z_b rows – do the following:*

   (a) *If possible, find a row referring to a basis variable among X_b to perform the pivot operation. This will maintain primal feasibility w.r.t. the variables Z_b.*

   (b) *If any basis variable X_j turns out to be negative, exclude X_j from basis and include M_j instead. This will be in accordance with the exclusion rule since index j does not belong to the set of indices for Z_b-variables.*

   (c) *If it is not possible to exclude an X_b-variable from the basis, exclude the appropriate M_b-variable and continue in the usual manner.*

   (d) *Once an X-variable has been omitted from the basis, exclude it from further consideration.*

4. Continue with the pivot operations in (2.) until all X-variables have been excluded.

Proof *See the Appendix.* ∎

Step 3b amounts to multiple shifts of basis variables. When steps 3a-c can be performed it is guaranteed that 1) a feasible solution is available and 2) the number of "artificial" X-variables decreases by *at least one*, but most likely by more than one and 3) the exclusion rule is obeyed at all points in time. Once all X-variables have been omitted from the basis the optimal solution called for is obtained. However, it may be necessary to perform pivot operations of the form "include one Z_n-variable and exclude another Z_b-variable". It depends upon the structure of the problem how frequent – if at all – the algorithm will plunge into exploiting the possibility of multiple change of basis variables.

The algorithm described is perfectly general, except that the phenomenon of cycling is not discussed explicitly. We do not explicitly incorporate this in the description given above. It is known to be an extremely rare phenomenon in general and methods exist to overcome this problem.

In the following sections some special structure on the tableaus shown is established by postulating some special structure on $\boldsymbol{\Omega}$.

3 Special structure of the covariance matrix

Consider a $N \times N$ covariance matrix of the following form:

$$\boldsymbol{\Omega} = \boldsymbol{D} + \boldsymbol{B}\boldsymbol{\Sigma}\boldsymbol{B}^{\boldsymbol{t}} \tag{3.1}$$

where

- $\boldsymbol{D}$ is a $N \times N$ diagonal covariance matrix representing "unsystematic risk"
- $\boldsymbol{\Sigma}$ is a $K \times K$ matrix representing the covariance matrix of the "systematic risk factors" and
- $\boldsymbol{B}$ is a $N \times K$ matrix of "factor loadings" b_{jk}

This covariance structure arises from the following return generating processes for asset returns:

$$R_j = \overline{R}_j + \sum_{k=1}^{K} b_{kj}\eta_k + e_j \qquad j = 1, 2, \ldots, N \tag{3.2}$$

with the usual interpretation:

- $\overline{R}_j$ is the expected return on asset j
- b_{kj} is the response of asset j to the k'th "systematic risk factor" η_k
- e_j is the "residual" or "unsystematic risk factor" of asset j.

By assumption the residual risk factors are mutually independent random variables and also independent of the systematic risk factors. Note, however, that it is not possible from the covariance matrix itself to give any economic interpretation of the specification of the factors. If matrix $\boldsymbol{B}$ is in accordance with the covariance structure, so is the matrix $-\boldsymbol{B}$.

For the type of matrices in (3.1) the inverse is easily found[8] by means of a more general matrix inversion lemma.

Lemma 1 *Let the $N \times N$-matrix $\boldsymbol{F}$ be given as*

$$\boldsymbol{F} \equiv \boldsymbol{G} + \boldsymbol{H}\boldsymbol{M}\boldsymbol{H}^{\boldsymbol{t}} \tag{3.3}$$

where

- $\boldsymbol{G}$ *is a symmetric $N \times N$ matrix*
- $\boldsymbol{M}$ *is a $K \times K$ matrix, $K \leq N$ and*
- $\boldsymbol{H}$ *is a $N \times K$ matrix*

Provided $\boldsymbol{G}$ as well as $\boldsymbol{M}$ are non-singular matrices the matrix $\boldsymbol{F}$ is also non-singular with the inverse matrix

$$\boldsymbol{F}^{-1} = \boldsymbol{G}^{-1} - \boldsymbol{G}^{-1}\boldsymbol{H}\left[\boldsymbol{M}^{-1} + \boldsymbol{H}^{t}\boldsymbol{G}^{-1}\boldsymbol{H}\right]^{-1}\boldsymbol{H}^{t}\boldsymbol{G}^{-1} \tag{3.4}$$

Proof *This can be proved by verification in a straightforward manner. Multiply the candidate given in (3.4) for the inverse with $\boldsymbol{F}$ and reduce the expressions.* ∎

Applying this matrix inversion lemma to the covariance matrix in (3.1) results in

$$\boldsymbol{\Omega}^{-1} = \boldsymbol{D}^{-1} - \boldsymbol{D}^{-1}\boldsymbol{B}\left[\boldsymbol{\Sigma}^{-1} + \boldsymbol{B}^{t}\boldsymbol{D}^{-1}\boldsymbol{B}\right]^{-1}\boldsymbol{B}^{t}\boldsymbol{D}^{-1} \tag{3.5}$$

Calculation of this inverse involves a trivial calculation of the inverse of a diagonal matrix $\boldsymbol{D}$. Furthermore, the covariance matrices $\boldsymbol{\Sigma}$ as well as $\boldsymbol{\Sigma}^{-1} + \boldsymbol{B}^{t}\boldsymbol{D}^{-1}\boldsymbol{B}$ must be inverted. However, the assumption behind and usefulness of the factor or APT specification in (3.1) is that $K << N$. This means that the computational burden of inverting the $N \times N$-matrix $\boldsymbol{\Omega}$ is reduced to that of inverting a $K \times K$-matrix. As a matter of fact this can be done with no direct inversion at all in K simple steps, cf. Kwan (1984).

Solving the first order conditions (2.2) above for the *unrestricted* case we have

$$\boldsymbol{Z} = \boldsymbol{\Omega}^{-1}(\overline{\boldsymbol{R}} - R_f\mathbf{1}) =$$

$$\boldsymbol{D}^{-1}(\,\overline{\boldsymbol{R}} - R_f\mathbf{1}\,) - \boldsymbol{D}^{-1}\boldsymbol{B}[\,\boldsymbol{\Sigma}^{-1} + \boldsymbol{B}^{t}\boldsymbol{D}^{-1}\boldsymbol{B}\,]^{-1}\,\boldsymbol{B}^{t}\boldsymbol{D}^{-1}\,(\overline{\boldsymbol{R}} - R_f\mathbf{1}\,) \tag{3.6}$$

or

$$\boldsymbol{Z} = \boldsymbol{D}^{-1}(\,\overline{\boldsymbol{R}} - R_f\mathbf{1}\,) - \boldsymbol{D}^{-1}\boldsymbol{B}\boldsymbol{C} \tag{3.7}$$

where $\boldsymbol{C}$ is the K-vector

$$\boldsymbol{C} = [\,\boldsymbol{\Sigma}^{-1} + \boldsymbol{B}^{t}\boldsymbol{D}^{-1}\boldsymbol{B}\,]^{-1}\,\boldsymbol{B}^{t}\boldsymbol{D}^{-1}(\,\overline{\boldsymbol{R}} - R_f\mathbf{1}\,) \tag{3.8}$$

The matrix inversion lemma applies to submatrices of $\boldsymbol{\Omega}$ as well, relevant for portfolios formed from a subset of the entire menu of assets. In the next sections we will reconcile the reduced form expressions for some standard cases found in the literature with our linear programming approach.

[8] We assume that no index portfolio exists so that the matrix $\boldsymbol{D}$ is singular.

4 The single index model

The single index model is a *one factor model*, specified by the following assumption:[9]

$$R_j = \alpha_j + \beta_j R_m + e_j, \quad e_j \perp e_i \ , e_j \perp R_m \ \text{ for } i \neq j \tag{4.1}$$

As a consequence of this assumption we have the relations

$$\sigma_j^2 = \beta_j^2 \sigma_m^2 + \sigma_{e_j}^2 \tag{4.2}$$

$$\sigma_{ij} = \beta_i \beta_j \sigma_m^2 \tag{4.3}$$

In terms of the notation of section 3 this can be written as

$$\boldsymbol{\Sigma} \equiv \sigma_m^2, \qquad \boldsymbol{B} \equiv \boldsymbol{\beta}, \qquad \boldsymbol{D} \equiv \operatorname{diag}\left(\sigma_{e_1}^2, \sigma_{e_2}^2, \ldots, \sigma_{e_N}^2\right) \tag{4.4}$$

In this special case the optimal $\boldsymbol{Z}$ is given by $\boldsymbol{\Theta}_0 - \boldsymbol{\Theta}_1$, where

$$\boldsymbol{\Theta}_0 = \boldsymbol{D^{-1}} \left(\overline{\boldsymbol{R}} - R_f \mathbf{1} \right) = \begin{pmatrix} \frac{\overline{R}_1 - R_f}{\sigma_{e_1}^2} \\ \frac{\overline{R}_2 - R_f}{\sigma_{e_2}^2} \\ \vdots \\ \frac{\overline{R}_N - R_f}{\sigma_{e_N}^2} \end{pmatrix} \tag{4.5}$$

and

$$\boldsymbol{\Theta}_1 = \boldsymbol{D^{-1}} \boldsymbol{\beta} C \tag{4.6}$$

As for C in (3.8) we have $C = \Phi \boldsymbol{\beta^t} \boldsymbol{\Theta}_0$, where

$$\boldsymbol{\beta^t} \boldsymbol{\Theta}_0 = \boldsymbol{\beta^t} \boldsymbol{D^{-1}} \left(\overline{\boldsymbol{R}} - R_f \mathbf{1} \right) = \sum_{j=1}^{n} \frac{\beta_j}{\sigma_{e_j}^2} \cdot (\overline{R}_j - R_f) \tag{4.7}$$

$$\Phi \equiv \left[\boldsymbol{\Sigma^{-1}} + \boldsymbol{B^t} \boldsymbol{D^{-1}} \boldsymbol{B} \right]^{-1} = \frac{1}{\frac{1}{\sigma_m^2} + \boldsymbol{\beta^t} \boldsymbol{D^{-1}} \boldsymbol{\beta}} = \frac{\sigma_m^2}{1 + \sigma_m^2 \sum_{j=1}^{N} \frac{\beta_j^2}{\sigma_{e_j}^2}} \tag{4.8}$$

and

$$\boldsymbol{D^{-1}} \boldsymbol{\beta} = \left(\beta_1 / \sigma_{e_1}^2, \beta_2 / \sigma_{e_2}^2, \ldots, \beta_N / \sigma_{e_N}^2 \right)^t \tag{4.9}$$

That is,

$$\boldsymbol{Z} = \begin{pmatrix} \frac{\overline{R}_1 - R_f - \beta_1 C}{\sigma_{e_1}^2} \\ \frac{\overline{R}_2 - R_f - \beta_2 C}{\sigma_{e_2}^2} \\ \vdots \\ \frac{\overline{R}_N - R_f - \beta_N C}{\sigma_{e_N}^2} \end{pmatrix} \tag{4.10}$$

[9] A less restrictive set of assumptions excludes the requirement $e_j \perp e_i$, and the model is called the market model. See e.g. Elton and Gruber (1995), p. 152.

The same model with short sale constraints is a special case of the general model described in section 2. If an asset h exists with $\beta_h = 0$ the last term in the optimal value for Z_h will vanish and the only criterion for optimality is that $\overline{R}_h - R_f > 0$. This is so because the row and column corresponding to this variable will be proportional to the unit vector and unaffected by any subsequent change of basis. Hence such assets should be chosen as members of the initial basis, since they will remain with unchanged coefficients in all iterations. This means that computationally they need not be dealt with at all, except for the normalization of $\boldsymbol{Z}$.

For other assets, Elton, Gruber, and Padberg (1976) suggested that individual securities should be "ranked" according to their "Treynor ratio", and that this ranking should be used to determine the sequence in which assets should be included in the optimal portfolio. The ranking by the Treynor ratio works as follows:

- positive β-securities in *decreasing* order of $\frac{\overline{R}_i - R_f}{\beta_i}$ and
- negative β-securities in *increasing* order of $\frac{\overline{R}_i - R_f}{\beta_i}$

Elton, Gruber, and Padberg (1976) demonstrated that while the coefficient C would change in this process, their selection procedure based on the Treynor ratio would select securities one by one, never omit an already included security and come to an end where the optimal portfolio is composed of

- all positive β-securities with a Treynor ratio above the endogenously determined "cut-off" value of C would be included
- all negative β-securities with a Treynor ratio below the endogenously determined "cut-off" value of C would be included

Putting this into our linear programming framework the algorithm will stop exactly after a number of iterations equal to the number of securities in the optimal portfolio. The original algorithm is one special case of the linear programming routine where the sequence of in-coming basis variables is chosen in a specific way without ever violating the exclusion rule. Assuming this ordering of in-coming basis variables and looking away from zero-β assets it is certain that a security that has entered into the basis at some iteration will never leave again. At the end the Lagrangian multipliers for any excluded asset p can be read off the tableau as $C_b\beta_p - (\overline{R}_p - R_f)$, where subscript b refers to the actual set of basis variables – i.e. the securities in the optimal portfolio – when the algorithm stops.

The details of these considerations are slightly tedious in terms of technical notation. Hence, they are carried out in the Appendix.

5 The constant correlation model

In the constant correlation case the covariance matrix is

$$\boldsymbol{\Omega} = \begin{bmatrix} \sigma_1^2 & \dots & \rho\sigma_1\sigma_j & \dots & \rho\sigma_1\sigma_N \\ \rho\sigma_2\sigma_1 & \sigma_2^2 & \dots & \dots & \rho\sigma_2\sigma_N \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \rho\sigma_1\sigma_N & \rho\sigma_2\sigma_N & \dots & \dots & \sigma_N^2 \end{bmatrix} =$$

$$(1-\rho)\begin{bmatrix} \sigma_1^2 & 0 & \dots & & 0 \\ 0 & \sigma_2^2 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & 0 & \sigma_N^2 \end{bmatrix} + \boldsymbol{\sigma}\rho\boldsymbol{\sigma^t} \tag{5.1}$$

where

$$\boldsymbol{\sigma^t} = (\sigma_1, \sigma_2, \dots, \sigma_N) \tag{5.2}$$

Provided $0<\rho<1$ this model has a covariance structure that is mathematically equivalent to a special case of the single index model with "unsystematic" risk $(1-\rho)\sigma_j^2$.[10] It is convenient to write this as $\widetilde{\boldsymbol{D}} \equiv (1-\rho)\boldsymbol{D}$. Furthermore, $\boldsymbol{\Sigma} \equiv \rho$ and $\boldsymbol{\beta} \equiv \boldsymbol{\sigma}$. Hence

$$\boldsymbol{\Omega^{-1}} = \widetilde{\boldsymbol{D}}^{\boldsymbol{-1}} - \widetilde{\boldsymbol{D}}^{\boldsymbol{-1}}\boldsymbol{\sigma}\left[\frac{1}{\rho} + \boldsymbol{\sigma^t}\widetilde{\boldsymbol{D}}^{\boldsymbol{-1}}\boldsymbol{\sigma}\right]^{-1}\boldsymbol{\sigma^t}\widetilde{\boldsymbol{D}}^{\boldsymbol{-1}} \tag{5.3}$$

By construction we notice the relation $\boldsymbol{\sigma^t}\widetilde{\boldsymbol{D}}^{\boldsymbol{-1}}\boldsymbol{\sigma} = \frac{N}{1-\rho}$. With a little manipulation the solution for the optimal Z in the case with no short sale constraints becomes

$$Z_i = \frac{1}{1-\rho}\left[\frac{\overline{R}_i - R_f}{\sigma_i^2} - \frac{\rho}{1-\rho+\rho N}\frac{1}{\sigma_i^2}\sigma_i \sum_{j=1}^{N}\frac{\overline{R}_j - R_f}{\sigma_j}\right] =$$

$$\frac{1}{(1-\rho)\sigma_i}\left[\frac{\overline{R}_i - R_f}{\sigma_i} - \frac{\rho}{1-\rho+\rho N}\sum_{j=1}^{N}\frac{\overline{R}_j - R_f}{\sigma_j}\right] \tag{5.4}$$

The selection procedure in case of short sale constraints is simple in this case, since it falls into the "only positive β-value" category. Hence rank securities by their "Sharpe ratio" $\frac{\overline{R}_j - R_f}{\sigma_j}$ and include them sequentially until the "cut-off"-value of C_b occurs. It is straightforward to see that C_b has the form

$$C_b = \frac{\rho}{1-\rho+\rho k}\sum_{j=1}^{k}\frac{R_j - R_f}{\sigma_j} \tag{5.5}$$

Example 1

A simple example with 3 assets is shown here. The assets have the same standard deviation ($\sigma_i = 1$), and $\rho = 0.5$. The Sharpe ratios, which are identical in this case to the excess expected return, is given by the vector (10,4,2). Hence the assets are ordered according to a decreasing Sharpe ratio.

The *usual* simplex routine goes through the following sequence of steps:

1. The initial basis is $(X_1, X_2, X_3) = (10, 4, 2)$, and the value of the objective function is 16.
2. The entering variable is chosen as Z_1. The exiting variable is X_3.
3. The new basis is $(Z_1, X_1, X_2) = (4, 6, 2)$, and the value of the objective function is 8.

[10] However, the "residuals" e_i are *not* residuals, but *total* variances of individual assets. Hence the assumption of orthogonality of the e_i does not apply.

4. The entering variable is chosen as M_3. The exiting variable is X_2.

5. The new basis is $(Z_1, M_3, X_1) = (8, 2, 2)$, and the value of the objective function is 2.

6. The entering variable is chosen as M_2. The exiting variable is X_1.

7. The new basis is $(M_2, M_3, Z_1) = (1, 3, 10)$, and the value of the objective function is 0.

8. The algorithm is finished. The cut-off rate C_b is 5 in accordance with the Lagrangian multipliers being 1 and 3, respectively.

	X_0	Z_1	Z_2	Z_3	M_1	M_2	M_3	X_1	X_2	X_3	rhs
X_0	1	2	2	2	-1	-1	-1	0	0	0	16
X_1	0	1	1/2	1/2	-1	0	0	1	0	0	10
X_2	0	1/2	1	1/2	0	-1	0	0	1	0	4
X_3	0	1/2	1/2	1	0	0	-1	0	0	1	2
X_0	1	0	0	-2	-1	-1	3	0	0	-4	8
X_1	0	0	$-1/2$	$-3/2$	-1	0	2	1	0	-2	6
X_2	0	0	1/2	$-1/2$	0	-1	1	0	1	-1	2
Z_1	0	1	1	2	0	0	-2	0	0	2	4
X_0	1	0	$-3/2$	$-1/2$	-1	2	0	0	-3	-1	2
X_1	0	0	$-3/2$	$-1/2$	-1	2	0	1	-2	0	2
M_3	0	0	1/2	$-1/2$	0	-1	1	0	1	-1	2
Z_1	0	1	2	1	0	-2	0	0	2	0	8
X_0	1	0	0	0	0	0	0	-1	-1	-1	0
M_2	0	0	$-3/4$	$-1/4$	$-1/2$	1	0	1/2	-1	0	1
M_3	0	0	$-1/4$	$-3/4$	$-1/2$	0	1	1/2	0	-1	3
Z_1	0	1	1/2	1/2	-1	0	0	1	0	0	10

Table 1: Ordinary simplex iterations for numerical example 1

Example 2

Alternatively we show how to use the algorithm in theorem 1 and the possibility of multiple basis shifts. Furthermore, by exploiting the ranking device for this example the need for pivoting steps of the form "include one Z_n-variable and exclude another Z_b-variable" is eliminated. The algorithm in theorem 1 goes through the following steps:

1. The initial basis is $(X_1, X_2, X_3) = (10, 4, 2)$, and the value of the objective function is 16.

2. The entering variable is chosen as Z_1. The exiting variable is X_1.

3. The new basis is $(Z_1, X_2, X_3) = (10, -1, -3)$, and the value of the objective function is -4.

4. Primal feasibility is re-established by exchanging $(X_2, X_3) = (-1, -3)$ with $(M_2, M_3) = (1, 3)$.

5. The algorithm is finished. The cut-off rate is 5 in accordance with the Lagrangian multipliers being 1 and 3, respectively.

In step 2 the exiting variable is chosen as X_1, since no Z_b-variable will become negative by excluding further search for a pivot element. However, the two other basis variables, X_2 and X_3 become negative. This is fixed in step 4 through a multiple basis shift.

	X_0	Z_1	Z_2	Z_3	M_1	M_2	M_3	X_1	X_2	X_3	rhs
X_0	1	2	2	2	-1	-1	-1	0	0	0	16
X_1	0	1	1/2	1/2	-1	0	0	1	0	0	10
X_2	0	1/2	1	1/2	0	-1	0	0	1	0	4
X_3	0	1/2	1/2	1	0	0	-1	0	0	1	2
X_0	1	0	5/4	5/4	1	-1	-1	-2	0	0	-4
Z_1	0	1	1/2	1/2	-1	0	0	1	0	0	10
X_2	0	0	3/4	1/4	1/2	-1	0	$-1/2$	1	0	-1
X_3	0	0	1/4	3/4	1/2	0	-1	$-1/2$	0	1	-3
X_0	1	0	0	0	0	0	0	-1	-1	-1	0
Z_1	0	1	1/2	1/2	-1	0	0	1	0	0	10
M_2	0	0	$-3/4$	$-1/4$	$-1/2$	1	0	1/2	-1	0	1
M_3	0	0	$-1/4$	$-3/4$	$-1/2$	0	1	1/2	0	-1	3

Table 2: Ordinary simplex iterations for numerical example 2

Given these parameters there is only one asset in the optimal portfolio. Hence, no real tangency occurs, since the slope of the restricted efficient frontier is too steep at the point representing the first asset. ∎

Performing a sensitivity analysis on the inputs reveals that when R_f is lowered by Δ, which means "sliding" down the efficient frontier, the optimal values will change as follows:

$$\begin{pmatrix} Z_1 \\ M_2 \\ M_3 \end{pmatrix} = \begin{pmatrix} 10+\Delta \\ 1-\Delta/2 \\ 3-\Delta/2 \end{pmatrix}$$

This implies that the solution with Z_1 alone will remain optimal until $\Delta = 2$. At that point Z_2 enters and M_2 leaves the basis.

When the basis variable M_2 has been exchanged for Z_2 the tableau looks as follows:

	X_0	Z_1	Z_2	Z_3	M_1	M_2	M_3	X_1	X_2	X_3	rhs
X_0	1	0	0	0	0	0	0	-1	-1	-1	0
Z_1	0	1	0	1/3	$-4/3$	2/3	0	4/3	$-2/3$	0	12
M_2	0	0	1	1/3	2/3	$-4/3$	0	$-2/3$	4/3	0	0
M_3	0	0	0	$-2/3$	$-1/3$	$-1/3$	1	1/3	1/3	-1	2

Table 3: Change of basis for numerical example 2 after reaching $\Delta = 2$.

Moving further down the efficient frontier we observe that the basis variables – after resetting Δ to 0 – change according to

$$\begin{pmatrix} Z_1 \\ Z_2 \\ M_3 \end{pmatrix} = \begin{pmatrix} 12 \\ 0 \\ 2 \end{pmatrix} + \Delta \begin{pmatrix} \frac{2}{3} \\ \frac{2}{3} \\ -\frac{1}{3} \end{pmatrix}$$

When this new Δ attains the value 6 the third asset comes into play. After another resetting of Δ to 0 the segment of the efficient frontier with all 3 assets have portfolio weights given by the normalized vector

$$\begin{pmatrix} Z_1 \\ Z_2 \\ Z_3 \end{pmatrix} = \begin{pmatrix} 16 \\ 4 \\ 0 \end{pmatrix} + \Delta \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$$

As $\Delta \to \infty$ the portfolio will converge to the minimum variance portfolio (1/3,1/3,1/3).

In this manner the LP-tableau enables an easy calculation of the entire restricted mean-variance efficient frontier once just one point has been located. This means that in order to trace out the entire efficient frontier only one linear programming problem has to be solved, i.e. only one point has to be solved for explicitly through an optimization problem. The rest follows by a simple sensitivity analysis on the optimal tableau resulting from this optimization.

6 The multi-group model

The multi-group model is an extended version of the constant correlation case. Instead of one correlation coefficient there is a number of groups, H, of securities within which the correlation among any two securities is identical. Additionally the correlation between any two securities in different groups is a constant depending on the groups, but not on the individual securities.

The structure of the covariance matrix is shown in detail below. For notational reasons we treat only the case with two groups. The first group has n_1 securities and the second has n_2 securities. We will also use the notation N_1 and N_2 for the set of indices relating to the securities in group no. 1 and group no. 2, respectively.

$$\begin{aligned} \boldsymbol{\Omega} &= \left(\boldsymbol{I} - \operatorname{diag} \left[\underbrace{\rho_{11}, \ldots, \rho_{11}}_{n_1 \ terms}, \underbrace{\rho_{22}, \ldots, \rho_{22}}_{n_2 \ terms} \right] \right) \begin{bmatrix} \sigma_1^2 & 0 & \ldots & & 0 \\ 0 & \sigma_2^2 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & \ldots & 0 & \sigma_N^2 \end{bmatrix} + \boldsymbol{B}\boldsymbol{\rho}\boldsymbol{B}^{\boldsymbol{t}} \\ &\equiv \widetilde{\boldsymbol{D}} + \boldsymbol{B}\boldsymbol{\rho}\boldsymbol{B}^{\boldsymbol{t}} \end{aligned} \tag{6.1}$$

where

$$\boldsymbol{B}^{\boldsymbol{t}} = \begin{bmatrix} \sigma_1, \ldots, \sigma_{n_1} & 0, \ldots, 0 \\ 0, \ldots, 0 & \sigma_{n_1+1}, \ldots, \sigma_N \end{bmatrix} \tag{6.2}$$

$$\boldsymbol{\rho} = \begin{bmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{bmatrix} \tag{6.3}$$

The inverse of this covariance matrix is

$$\boldsymbol{\Omega^{-1}} = \widetilde{\boldsymbol{D}}^{-1} - \widetilde{\boldsymbol{D}}^{-1}\boldsymbol{B}\,[\,\boldsymbol{\rho}^{-1} + \boldsymbol{B}^t\widetilde{\boldsymbol{D}}^{-1}\boldsymbol{B}\,]^{-1}\,\boldsymbol{B}^t\widetilde{\boldsymbol{D}}^{-1} \equiv \widetilde{\boldsymbol{D}}^{-1} - \widetilde{\boldsymbol{D}}^{-1}\boldsymbol{B}\boldsymbol{\Phi}\boldsymbol{B}^t\widetilde{\boldsymbol{D}}^{-1} \tag{6.4}$$

In the same manner as in the previous section it can be shown that the matrix $\boldsymbol{B}^t\widetilde{\boldsymbol{D}}^{-1}\boldsymbol{B}$ has a simple form. It is a diagonal matrix with k'th diagonal element $\frac{n_k}{1-\rho_{kk}}$. And working backwards in (6.4) we obtain

$$\boldsymbol{B}^t\widetilde{\boldsymbol{D}}^{-1}\left(\overline{\boldsymbol{R}} - R_f\mathbf{1}\right) = \mathbf{vec}\left(\sum_{i\in N_k}\frac{\overline{R}_i - R_f}{\sigma_i(1-\rho_{kk})}\right)_{k=1,\ldots,K} \tag{6.5}$$

For a security (=row) $i \in N_k$ we have

$$C_k \equiv \left(\boldsymbol{\Phi}\boldsymbol{B}^t\widetilde{\boldsymbol{D}}^{-1}(\,\overline{\boldsymbol{R}} - R_f\mathbf{1}\,)\right)_k = \sum_{j=1}^{K}\Phi_{kj}\left(\sum_{p\in N_j}\frac{\overline{R}_p - R_f}{\sigma_p(1-\rho_{jj})}\right) \tag{6.6}$$

The optimal portfolio weight for a security $i \in N_k$ thus has the form

$$Z_i = \frac{1}{\sigma_i(1-\rho_{kk})}\left[\frac{\overline{R}_i - R_f}{\sigma_i} - \Psi_k\right] \equiv \boldsymbol{e_i^t}\widetilde{\boldsymbol{D}}^{-1}\left(\overline{\boldsymbol{R}} - R_f\mathbf{1} - \boldsymbol{BC}\right) \tag{6.7}$$

As mentioned previously the computational burden involved in finding $\boldsymbol{\Psi}$ is modest, cf. the sequential procedure given in Kwan (1984).

If the groups were uncorrelated, i.e. $\rho_{kg} = 0$ for $k \neq g$, the problem with short sale constraints would be solved automatically. In that case $\boldsymbol{\Phi}$ is a diagonal matrix and the elements Φ_{kk} would be immediately calculable. The critical values of $\boldsymbol{C}$ – or equivalently $\boldsymbol{\Psi}$ – according to the procedure from the constant correlation model could then be determined within each group separately. However, due to the structure of correlation among groups this cannot be done sequentially.

A closer look at the structure of the simplex tableau at intermediate steps in the simplex algorithm reveals that ranking by the Sharpe ratio – as in the constant correlation model – is still a useful idea in the multi-group model. Whenever a particular asset in one group is part of the optimal solution, any asset in the same group with better ranking is also part of the optimal solution. Unfortunately, it is *not* possible to exclude the possibility – despite the ranking – that a security entered into the basis at one point of the iterations will be forced to leave at a later iteration because of the simultaneous changes at all coordinates of the "cut-off vector" $\boldsymbol{\Psi}$. However, it is not likely to happen, but in particular cases it does when running the simplex algorithm.[11]

Example 3

As a numerical example we perform the simplex iterations in our algorithm using the input values from the example given in Elton, Gruber, and Padberg (1977). The original problem had two groups with 8 and 7 assets, respectively. Given that the optimal portfolio has $n_1 = 3$ and $n_2 = 1$ we only include 4 assets in group 1 and 3 assets in group 2 in order to limit the space requirement for representing the calculations. This implies that the tableaus below do not

[11] In Elton, Gruber, and Padberg (1977) an appendix outlines a straightforward way of updating the vector Ψ, when a number of securities are added or excluded in the portfolio. The simplex routine will automatically do this in a simple manner.

demonstrate the amount of multiple basis changes that would take place if the entire menu of assets had been present.

The input data to this example are:

$$\boldsymbol{\rho} = \begin{pmatrix} \frac{1}{2} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{5} \end{pmatrix} \qquad \sigma_i = 1 \ \forall i \tag{6.8}$$

and the Sharpe ratios like given on the r.h.s. in the tableau.

Given the composition of the optimal portfolio, i.e. $n_1 = 3$ and $n_2 = 1$, we can calculate the matrix $\boldsymbol{\Phi}$:

$$\boldsymbol{\Phi} = \left[\boldsymbol{\rho}^{-1} + \begin{pmatrix} 6 & 0 \\ 0 & \frac{5}{3} \end{pmatrix}\right]^{-1} = \left[\begin{pmatrix} \frac{9}{2} & -\frac{15}{4} \\ -\frac{15}{4} & \frac{45}{8} \end{pmatrix} + \begin{pmatrix} 6 & 0 \\ 0 & \frac{5}{3} \end{pmatrix}\right]^{-1} = \begin{bmatrix} \frac{7}{60} & \frac{3}{50} \\ \frac{3}{50} & \frac{42}{250} \end{bmatrix} \tag{6.9}$$

Insert "Table 4"

Table 4: Change of basis for numerical example in Elton, Gruber, and Padberg (1977)

The vector $\boldsymbol{\Psi}$ is identical to the vector $\boldsymbol{C}$ in this example. We find $\boldsymbol{C}$ as follows:

$$\boldsymbol{C} = \boldsymbol{\Phi} \begin{pmatrix} 48 \\ \frac{40}{3} \end{pmatrix} = \begin{pmatrix} 6.4 \\ 5.12 \end{pmatrix} \tag{6.10}$$

This is in accordance with the findings in the last simplex tableau, where $m_4 = 0.4$ and $m_6 = 0.62$.[12]

7 Multi-index models

The single index model in section 4 is a special case of the more general multi-index or multi-factor model. The return generating mechanism for asset j is assumed driven by

1. one "market factor" R_m, common to all securities and
2. one sector specific factor, I_h, common to all securities in a given sector $h \in \{1, 2, \ldots, H\}$.
3. The factors are mutually uncorrelated and also uncorrelated with R_m, and a given sector specific factor will only contribute to "systematic risk" within that sector.

[12] Due to a miscalculation these numbers differ from the ones found in Elton, Gruber, and Padberg (1977), who used the correlation $\rho_{22} = \frac{2}{5}$ instead of $1 - \rho_{22} = \frac{3}{5}$ in their calculation of $\boldsymbol{\Phi}$.

The uncorrelatedness of the factors is not a restriction. The factors can always be "rotated" in order to fulfill this convenient assumption. In compact form the return generating mechanism can be stated as follows, analogous to (4.1) :

$$R_j = \alpha_j + \beta_j R_m + \sum_{h=1}^{H} b_{j,h} I_h + e_j \tag{7.1}$$

$$e_j \perp R_m,\ e_j \perp I_h\ \forall j,h,\ e_j \perp e_i \text{ for } i \neq j,\ I_h \perp R_m\ \forall h \tag{7.2}$$

The set of indices for securities in sector h is denoted N_h. With these assumptions the components in the covariance matrix are the following matrices, shown for convenience only for the case of two sectors (H=2) and with $N_1 = \{1, 2, \ldots, n_1\}$ and $N_2 = \{n_1+1, n_1+2, \ldots, n_1+n_2 = N\}$:

$$\boldsymbol{\Sigma} \equiv \operatorname{diag}[\sigma_m^2, \sigma_{I_1}^2, \sigma_{I_2}^2] \tag{7.3}$$

$$\boldsymbol{B} \equiv \begin{bmatrix} \beta_1 & b_{1,1} & 0 \\ \beta_2 & b_{2,1} & 0 \\ \vdots & \vdots & \vdots \\ \beta_{n_1} & b_{n_1,1} & 0 \\ \beta_{n_1+1} & 0 & b_{n_1+1,2} \\ \beta_{n_1+2} & 0 & b_{n_1+n_2,2} \\ \vdots & \vdots & \vdots \\ \beta_N & 0 & b_{N\,1,2} \end{bmatrix} \tag{7.4}$$

The optimal solution for Z in the unconstrained case derives from the general formula, cf. (3.7)-(3.8). Finding the $(H+1) \times 1$ vector C

$$\boldsymbol{C} = \left[\boldsymbol{\Sigma^{-1}} + \boldsymbol{B^t D^{-1} B}\right]^{-1} \boldsymbol{B^t D^{-1}} \left(\overline{\boldsymbol{R}} - R_f \mathbf{1}\right) \equiv \begin{pmatrix} C_0 \\ C_1 \\ \vdots \\ C_H \end{pmatrix} \tag{7.5}$$

involves an invertion of the matrix $[\boldsymbol{\Sigma^{-1}} + \boldsymbol{B^t D^{-1} B}]$. Given the constants C_p, $p = 0, 1, \ldots, H$, the optimal solution can be expressed as

$$Z_i = \frac{\overline{R}_i - R_f}{\sigma_{e_i}^2} - \frac{\beta_i}{\sigma_{e_i}^2} C_0 - \frac{b_{i,h}}{\sigma_{e_i}^2} C_h \tag{7.6}$$

where $i \in N_h$. Further details of these calculations are found in the Appendix.

The general form of the multi-index model can be handled by our linear programming procedure just as any other type of covariance matrix in order to find the efficient frontier under short sale constraints. However, since C_0 and C_h interact and any asset has both a β_j and a $b_{j,h}$ there is no short cut in the form of a ranking procedure.

In Elton, Gruber, and Padberg (1979) the model is re-interpreted in the sense that R_m is traded assset, i.e. it can be obtained as a portfolio of the assets specified. This leads to a degenerate matrix $\boldsymbol{D}$, because an asset exists with $\beta_m = 1$, $b_{m,h} = 0\ \forall h$ and no residual risk. The result of this is that the model – after suitable reformulation – falls into the category of the multi-group model in section 6 in its simplest form with no correlation between the groups. The reformulation is described in detail in the Appendix.

8 Including additional linear constraints

The linear programming algorithm is well suited to solve also problems with additional linear constraints. Typical constraints of this type are placement restrictions, e.g. of the form "maximum 40% of assets within a specific group" or "no individual asset may make up more than 20% of the entire portfolio value".

Example 4

We demonstrate this by a simple example and state a generalisation of theorem 1. We add to the above example 2 the following placement constraint:

"Assets no. 1 and 2 are not allowed to make up more than 50% of the entire portfolio value"

Solving for the Z-variables only determines the relative portfolio weights. To find the optimal portfolio the solution found must be normalized. Hence, the placement constraint must be formulated in relative terms as

$$Z_1 + Z_2 \leq 0.5(Z_1 + Z_2 + Z_3) \qquad \Leftrightarrow \qquad 0.5Z_1 + 0.5Z_2 - 0.5Z_3 + W = 0 \tag{8.1}$$

where $W \geq 0$ is an additional slack-variable.

Define the vector $\boldsymbol{a}$ by $\boldsymbol{a}^{\boldsymbol{t}} \equiv (0.5, 0.5, -0.5)$. Then the relevant first order conditions or Kuhn-Tucker conditions in this case become:

$$\begin{aligned}
\boldsymbol{\Omega Z} - \boldsymbol{M} + \boldsymbol{a}\mu &= \overline{\boldsymbol{R}} - R_f \mathbf{1} && (8.2)\\
\boldsymbol{a}^{\boldsymbol{t}}\boldsymbol{Z} + W &= 0 && (8.3)\\
Z_i, M_i &\geq 0 \quad i = 1, 2, \cdots, N, \quad W, \mu \geq 0 && (8.4)\\
Z_i M_i &= 0 \quad i = 1, 2, \cdots, N, \quad W\mu = 0 && (8.5)
\end{aligned}$$

This is fully equivalent with the setup in the former cases with no placement constraints. The same algorithm, making certain that W and μ are not simultaneously in the basis, will solve this problem. For reasons of space limitations we only record the final simplex tableau, but the steps in the algorithm are as follows:

1. The initial basis is $(X_1, X_2, X_3, W) = (10, 4, 2, 0)$, and the value of the objective function is 16.

2. The entering variable is chosen as Z_3. The exiting variable is X_3. It is necessary to include Z_3, because no feasible portfolio exists without this asset due to the placement constraint.

3. The new basis is $(Z_3, X_1, X_2, W) = (2, 9, 3, 1)$, and the value of the objective function is 12.

4. The entering variable is chosen as Z_1. The exiting variable is W.

5. The entering variable is μ. If the exiting variable is chosen as X_2 another ordinary simplex iteration with M_2 as entering and X_2 as exiting variable is necessary. If the exiting variable is chosen as X_1, which will not make any Z-variable negative, a multiple bias shift occurs. The value of X_2 will become negative, but X_2 can immediately be switched with M_2.

	X_0	Z_1	Z_2	Z_3	M_1	M_2	M_3	X_1	X_2	X_3	W	μ	rhs
X_0	1	0	0	0	0	0	0	-1	-1	-1	0	0	0
μ	0	0	$-1/2$	0	-1	0	1	1	0	-1	-1	1	8
M_2	0	0	$-11/12$	0	$-5/6$	1	$1/6$	$5/6$	-1	$-1/6$	$-1/2$	0	4
Z_3	0	0	$-1/6$	1	$-1/3$	0	$-1/3$	$1/3$	0	$1/3$	-1	0	4
Z_1	0	1	$5/6$	0	$-1/3$	0	$-1/3$	$1/3$	0	$1/3$	1	0	4

Table 5: Final simplex tableau for example 3 with placement constraints.

∎

Analogous to theorem 1 we conclude by stating the algorithm for the situation with additional linear constraints as theorem 2.

Theorem 2 *The following algorithm converges to the optimal solution of the modified linear programming problem in (2.6) with additional linear constraints:*

1. *Choose the initial basis as a combination of $M-$ and $X-$variables in accordance with the sign of the risk-premium. The initial basis variables for the rows corresponding to the additional linear constraints are chosen as the slack-variables W.*

2. *Select a current non-basis variable Z_i, M_i or μ_j to enter as a new basis variable. The usual criteria can be applied, i.e. looking for positive reduced cost coefficients, although the exclusion rules, $Z_i M_i = 0$ and $W_j \mu_j = 0$, must be obeyed.*

3. *Find the pivot element. If the entering variable is Z_i, check whether the pivot element refers to another Z_b-variable or to a $W-$variable. If so, perform the usual pivot operations. If not – and this is revealed the first time the usual pivot ratio in any row below the Z_b rows and the W rows is encountered to be lower than among the Z_b and W rows – do the following:*

   (a) *If possible, find a row referring to a basis variable among X_b to perform the pivot operation. This will maintain primal feasibility w.r.t. the variables Z_b and W_b.*

   (b) *If any basis variable X_j turns out to be negative, exclude X_j from basis and include M_j instead. This will be in accordance with the exclusion rule since index j does not belong to the set of indices for Z_b-variables.*

   (c) *If it is not possible to exclude an X_b-variable from the basis, exclude the appropriate M_b-variable or W_b-variable and continue in the usual manner.*

   (d) *Once an X-variable has been omitted from the basis it can be excluded from further consideration.*

4. *Continue with the pivot operations in (2.) until all X-variables have been excluded.*

Proof *See the Appendix.* ∎

9 Appendix

Proof [Theorem 1] *When the simplex algorithm comes to an end, we have the following three types of indices:*

1. $Z_j > 0,\ M_j = 0$
2. $Z_i = 0,\ M_i > 0$
3. $Z_k = M_k = 0$

Due to the exclusion rule these three subsets of indices are disjoint. Let

- $\tilde{\boldsymbol{\Omega}}$ *denote the subset of columns in* $\boldsymbol{\Omega}$ *relating to indices of type 1 and 3*
- $\tilde{\boldsymbol{Z}}$ *be the corresponding subset of* $\boldsymbol{Z}$
- $\widetilde{\boldsymbol{M}}$ *be the subset of columns in* $\boldsymbol{M}$ *relating to indices of type 2 and 3*

The final solution produced by the simplex algorithm with the exclusion rule solves the following LP-problem:

$$Min \sum_{p=1}^{N} X_p$$

$$subject\ to$$

$$\tilde{\boldsymbol{\Omega}}\tilde{\boldsymbol{Z}} - \widetilde{\boldsymbol{M}} + \boldsymbol{X} = \overline{\boldsymbol{R}} - R_f\mathbf{1}$$

$$\tilde{\boldsymbol{Z}},\ \widetilde{\boldsymbol{M}},\ \boldsymbol{X} \geq 0$$

Let the dual variables be denoted as $\boldsymbol{s}$*. Then*

$$\begin{aligned} s_j &\geq 0 \quad && \text{and } s_j = 0 \text{ whenever } \widetilde{M}_j > 0 \text{ for indices } j \text{ of type 2 and 3} \\ s_i &\leq 1 \quad && \text{and } s_i = 1 \text{ whenever } X_i > 0 \\ \boldsymbol{s^t}\tilde{\boldsymbol{\Omega}}\boldsymbol{e_j} &\leq 0 \quad && \text{and } \boldsymbol{s^t}\tilde{\boldsymbol{\Omega}}\boldsymbol{e_j} = 0 \text{ whenever } \tilde{Z}_j > 0 \text{ for indices } j \text{ of type 1 and 3} \end{aligned}$$

The "missing" columns in $\tilde{\boldsymbol{\Omega}}$ *correspond to indices for which* $\widetilde{M}_j > 0$*. We know that this guarantees both* $s_j = 0$ *and* $Z_j = 0$*. Hence we can conclude immediately that* $\boldsymbol{s^t\Omega s} = \boldsymbol{s^t}\tilde{\boldsymbol{\Omega}}\tilde{\boldsymbol{s}} = 0$*, where* $\tilde{\boldsymbol{s}}$ *is that part of* $\boldsymbol{s}$ *which corresponds to columns in* $\tilde{\boldsymbol{\Omega}}$*. Since* $\boldsymbol{\Omega}$ *is positive definite by assumption it also follows that* $\boldsymbol{s} = \mathbf{0}$*. Hence the optimal value of the dual problem is zero, which proves that the simplex algorithm comes to an end only after all X-variables have been eliminated.* ∎

Proof [Theorem 2] *When the simplex algorithm comes to an end, we have the following three types of indices concerning Z-variables:*

1. $Z_j > 0,\ M_j = 0$
2. $Z_i = 0,\ M_i > 0$
3. $Z_k = M_k = 0$

and the following three types of indices concerning W-variables:

4. $W_f > 0,\ \mu_f = 0$
5. $W_g = 0,\ \mu_g > 0$
6. $W_h = 0,\ \mu_h = 0$

Due to the exclusion rule both of these two groups of subsets of indices are mutually disjoint. Let

- $\widetilde{\boldsymbol{\Omega}}$ *denote the subset of columns in* $\boldsymbol{\Omega}$ *relating to indices of type 1 and 3*
- $\widetilde{\boldsymbol{Z}}$ *be the corresponding subset of* $\boldsymbol{Z}$
- $\widetilde{\boldsymbol{M}}$ *be the subset of columns in* $\boldsymbol{M}$ *relating to indices of type 2 and 3*
- $\widetilde{\boldsymbol{A}}$ *denote the subset of columns in* $\boldsymbol{A}$ *relating to indices of type 5 and 6*
- $\widetilde{\boldsymbol{\mu}}$ *denote the corresponding subset of* $\widetilde{\boldsymbol{\mu}}$
- $\widehat{\boldsymbol{A}}$ *denote the subset of rows in* $\widetilde{\boldsymbol{A}}$ *relating to indices of type 1 and 3*

The final solution produced by the simplex algorithm with the exclusion rule solves the following LP-problem:

$$Min \sum_{p=1}^{N} X_p$$

subject to

$$\begin{aligned} \widetilde{\boldsymbol{\Omega}}\widetilde{\boldsymbol{Z}} - \widetilde{\boldsymbol{M}} + \boldsymbol{X} + \widetilde{\boldsymbol{A}}\widetilde{\boldsymbol{\mu}} &= \overline{\boldsymbol{R}} - R_f\mathbf{1} \\ \widehat{\boldsymbol{A}}^{\boldsymbol{t}}\widetilde{\boldsymbol{Z}} + \widetilde{\boldsymbol{I}}\widetilde{\boldsymbol{W}} &= \mathbf{0} \end{aligned}$$

$$\widetilde{\boldsymbol{Z}},\ \widetilde{\boldsymbol{M}},\ \widetilde{\boldsymbol{W}}, \boldsymbol{X},\ \widetilde{\boldsymbol{\mu}} \geq 0$$

Let the dual variables associated with the $\widetilde{\boldsymbol{\Omega}}$*-rows be denoted by* $\boldsymbol{s}$ *and the dual variables associated with the* $\widehat{\boldsymbol{A}}$*-rows be denoted by* $\boldsymbol{q}$*. Then*

$$\begin{aligned} s_j \geq 0 \quad & \textit{and } s_j = 0 \textit{ whenever } \widetilde{M_j} > 0 \textit{ for indices } j \textit{ of type 2 and 3} \\ s_i \leq 1 \quad & \textit{and } s_i = 1 \textit{ whenever } X_i > 0 \\ \boldsymbol{s^t\widetilde{A}e_j} \leq 0 \quad & \textit{and } \boldsymbol{s^t\widetilde{A}e_j} = 0 \textit{ whenever } \widetilde{\mu}_j > 0 \textit{ for indices } j \textit{ of type 5 and 6} \\ q_p \leq 0 \quad & \textit{and } q_p = 0 \textit{ whenever } \widetilde{W}_j > 0 \textit{ for indices of type 5 and 6} \\ (\boldsymbol{s^t\widetilde{\Omega} + q^t\widehat{A}^t})\boldsymbol{e_j} \leq 0 \quad & \textit{and } (\boldsymbol{s^t\widetilde{\Omega} + q^t\widehat{A}^t})\boldsymbol{e_j} = 0 \textit{ whenever } \widetilde{Z}_j > 0 \textit{ for indices of type 1 and 3} \end{aligned}$$

As in the proof of theorem 1 we can conclude that

$$\boldsymbol{s^t\widetilde{\Omega}\widetilde{s} + q^t\widehat{A}^t\widetilde{s} = s^t\Omega s + q^t\widehat{A}^t\widetilde{s}} \leq 0$$

Since $\widehat{\boldsymbol{A}^t}\widetilde{\boldsymbol{s}}$ as well as $\boldsymbol{q}$ have non-positive components we also know that the last part of this sum is non-negative. Due to the positive definiteness of $\boldsymbol{\Omega}$ we conclude. like in the proof of theorem 2, that $\boldsymbol{s}$=0. Hence, all of the artificial variables X_i vanish.

Since this does not depend on the rhs of the problem it is also the case that the linear programming algorithm solves the problem with other linear restrictions. ∎

The single index model (section 4)

As the simplest example consider the case where all assets have positive β-values. Assume that assets have been sorted in order of decreasing value of their Treynor-ratio as described in section 4. Define Φ_k as Φ_b, where $b = \{1, 2, \ldots, k\}$. Define C_k analogously. Then it is straightforward to see that

$$C_k = C_{k-1}\frac{\Phi_k}{\Phi_{k-1}} + \left(1 - \frac{\Phi_k}{\Phi_{k-1}}\right)\frac{\overline{R}_k - R_f}{\beta_k} \tag{9.1}$$

The sequence Φ_k is a *decreasing* sequence. For this reason the sequence C_k has by construction the "single peak property" and hence a uniquely determined maximum. Each new value of C_k is a *convex combination* of its own former value C_{k-1} and a positive number: $\frac{\overline{R}_k - R_f}{\beta_k}$. The sequence will increase until the point where the entering number – $\frac{\overline{R}_k - R_f}{\beta_k}$ – becomes lower than the accumulated number C_{k-1}. After that point it will start to decline. Formally this can proved as follows.

Assume that $C_k < C_{k-1}$. This can only happen when $\frac{\overline{R}_k - R_f}{\beta_k} < C_k < C_{k-1}$. Since C_{k+1} is a convex combination of C_k and $\frac{\overline{R}_{k+1} - R_f}{\beta_{k+1}}$ ($< \frac{\overline{R}_k - R_f}{\beta_k} < C_k$) it must be the case that $C_{k+1} < C_k$. Hence, if the sequence C_k is observed to decline once it will continue to decline. That proves the single peak property.

Given that

- the assets have been numbered according to the ranking by their Treynor ratio
- the assets are entered into the basis in that sequence and that the pivot element in each step is chosen as the diagonal element in the covariance matrix

we can apply the matrix inversion lemma to the appropriate subset of assets in the basis:

$$\boldsymbol{\Omega}_{\boldsymbol{bb}}^{-\mathbf{1}} = \boldsymbol{D}_{\boldsymbol{b}}^{-\mathbf{1}}\left[\boldsymbol{I} - \Phi_b[\boldsymbol{\beta}_{\boldsymbol{b}}\boldsymbol{\beta}_{\boldsymbol{b}}^{\boldsymbol{t}}]\,\boldsymbol{D}_{\boldsymbol{b}}^{-\mathbf{1}}\right] \tag{9.2}$$

$$\Phi_b = \frac{\sigma_m^2}{1 + \sigma_m^2\boldsymbol{\beta}_{\boldsymbol{b}}^{\boldsymbol{t}}\boldsymbol{D}_{\boldsymbol{b}}^{-\mathbf{1}}\boldsymbol{\beta}_{\boldsymbol{b}}} = \frac{\sigma_m^2}{1 + \sigma_m^2\sum_{i\in b}\frac{\beta_i^2}{\sigma_{e_i}^2}} \quad \Rightarrow \tag{9.3}$$

$$\Phi_b = \sigma_m^2\left[1 - \Phi_b\left[\boldsymbol{\beta}_{\boldsymbol{b}}^{\boldsymbol{t}}\boldsymbol{D}_{\boldsymbol{b}}^{-\mathbf{1}}\boldsymbol{\beta}_{\boldsymbol{b}}\right]\right] \tag{9.4}$$

The coefficients in the inverted tableau at some intermediate step become:

X_0	$\boldsymbol{Z_b}$	$\boldsymbol{Z_n}$	$\boldsymbol{M}$
1	$\mathbf{0^t}$	$\mathbf{1_x^t}\left[\boldsymbol{D_n}+\Phi_b\boldsymbol{\beta_n}\boldsymbol{\beta_n^t}\right]$	$\Phi_b\mathbf{1_x^t}\boldsymbol{\beta_n}\boldsymbol{\beta_b^t}\boldsymbol{D_b^{-1}}\quad -\mathbf{1_x^t}$
$\mathbf{0}$	$\boldsymbol{I}$ $\mathbf{0}$	$\Phi_b\boldsymbol{D_b^{-1}}\boldsymbol{\beta_b}\boldsymbol{\beta_n^t}$ $\boldsymbol{D_n}+\Phi_b\boldsymbol{\beta_n}\boldsymbol{\beta_n^t}$	$-\boldsymbol{\Omega_{bb}^{-1}}\quad \mathbf{0}$ $\Phi_b\boldsymbol{\beta_n}\boldsymbol{\beta_b^t}\boldsymbol{D_b^{-1}}\quad -\boldsymbol{I}$

$\boldsymbol{X}$	rhs
$-\mathbf{1^t}-\Phi_b\mathbf{1_x^t}\boldsymbol{\beta_n}\boldsymbol{\beta_b^t}\boldsymbol{D_b^{-1}}\quad -\mathbf{1_{n_x}^t}$	$\mathbf{1_x^t}\left[\overline{\boldsymbol{R}}_x-R_f\mathbf{1_x}-\Phi_b\beta_n\beta_b^tD_b^{-1}\left(\overline{\boldsymbol{R}}_b-R_f\mathbf{1_b}\right)\right]$
$\boldsymbol{\Omega_{bb}^{-1}}\quad \mathbf{0}$ $-\Phi_b\boldsymbol{\beta_n}\boldsymbol{\beta_b^t}\boldsymbol{D_b^{-1}}\quad \boldsymbol{I}$	$\begin{pmatrix}\boldsymbol{\Omega_{bb}^{-1}} & \mathbf{0}\\ -\Phi_b\boldsymbol{\beta_n}\boldsymbol{\beta_b^t}\boldsymbol{D_b^{-1}} & \boldsymbol{I}\end{pmatrix}\left(\overline{\boldsymbol{R}}-R_f\mathbf{1}\right)$

At any iteration the right hand side will look as follows:

$$\begin{pmatrix}\vdots\\ \frac{\beta_i}{\sigma_{\varepsilon_i}^2}\left[\frac{\overline{R}_i-R_f}{\beta_i}-C_b\right]\\ \vdots\\ \overline{R}_j-R_f-\beta_jC_b\\ \vdots\end{pmatrix}$$

It is easy to see that pivoting "on the diagonal" will not destroy primal fasibility for assets previously entered. However, there may well be primal infeasibility in rows further down in the tableau. This is easily repaired by switching the basis entry to the corresponding Lagrangian multiplier variable. The algorithm will stop exactly when the single peak of C_b has been obtained.

As mentioned in section 3 the matrix $\boldsymbol{B}$ does not have an economic interpretation in its own. Hence the mirror case with pure negative β-values is a representation of the same validity. In that case the sequence C_k has the "single through" property, but the arguments are entirely analogous.

In the general case, with both positive and negative β-values, it becomes slightly more involved to prove convergence of the algorithm. The sequence C_k is now formed by including assets according to the sequence in which they are ordered within the two groups of positive and negative β-values, respectively, and it becomes necessary to switch between the two types of assets whenever a *local* peak/through is reached. However, any value for C_k will always be between the most recently recorded local maximum and local minimum, whenever the ordering within each of the groups is obeyed. The simplex method could be applied to keep track of this in a straightforward manner and to guarantee convergence without further qualification, since it is a general procedure.

The group of excluded assets – apart from the zero-β-assets – also falls into two groups.

1. If $\beta_j > 0$ the complementary slackness condition is satisfied:

$$\beta_j C_b - (R_j - R_f) > 0 \quad \Leftrightarrow \quad C_b > \frac{R_j - R_f}{\beta_j}$$

   Otherwise the asset would automatically have been added by the selection procedure, because C_b could be increased.

2. If $\beta_j < 0$ the complementary slackness condition is satisfied:

$$\beta_j C_b - (R_j - R_f) > 0 \quad \Leftrightarrow \quad C_b < \frac{R_j - R_f}{\beta_j}$$

   Otherwise the asset would automatically have been added by the selection procedure, because C_b could have been decreased.

The multi-index model (section 7)

$\boldsymbol{\Sigma}$ is already in diagonal form, and due to the block-structure of matrix $\boldsymbol{B}$ the matrix $\boldsymbol{B^t D^{-1} B}$ is almost a diagonal matrix. Only the first row and the first column has non-zero off-diagonal components.

$$\boldsymbol{\Sigma^{-1} + B^t D^{-1} B} = \boldsymbol{\Psi} + \begin{pmatrix} 0 \\ \boldsymbol{g} \end{pmatrix} (1,0,\ldots,0) + (1,0,\ldots,0)^t \, (0, \boldsymbol{g^t}) \tag{9.5}$$

$$\boldsymbol{\Psi} = \operatorname{diag}\left[\frac{1 + \sigma_m^2 \sum_{j=1}^{N} \frac{\beta_j^2}{\sigma_{e_j}^2}}{\sigma_m^2}, \frac{1 + \sigma_{I_1}^2 \sum_{j \in N_1} \frac{b_{j,1}^2}{\sigma_{e_j}^2}}{\sigma_{I_1}^2}, \ldots, \frac{1 + \sigma_{I_H}^2 \sum_{j \in N_H} \frac{b_{j,H}^2}{\sigma_{e_j}^2}}{\sigma_{I_H}^2} \right] \tag{9.6}$$

$$\boldsymbol{g^t} = \left[\sum_{j \in N_1} \frac{\beta_j b_{j,1}}{\sigma_{e_j}^2}, \ldots, \sum_{j \in N_H} \frac{\beta_j b_{j,H}}{\sigma_{e_j}^2} \right] \tag{9.7}$$

This matrix falls into the category of partitioned matrices for which inversion formulas are readily available:

$$[\boldsymbol{\Sigma^{-1} + B^t D^{-1} B}]^{-1} \equiv \begin{bmatrix} \Psi_1 & \boldsymbol{g^t} \\ \boldsymbol{g} & \boldsymbol{\Psi_2} \end{bmatrix}^{-1} = \begin{bmatrix} a & -a\boldsymbol{g^t \Psi_2^{-1}} \\ -a\boldsymbol{\Psi_2^{-1} g} & \boldsymbol{\Psi_2^{-1}} + a\boldsymbol{\Psi_2^{-1} g g^t \Psi_2^{-1}} \end{bmatrix} \tag{9.8}$$

where

$$a = \frac{1}{\Psi_1 - \boldsymbol{g^t \Psi_2^{-1} g}}$$

Hence

$$C_0 = a \sum_{j=1}^{N} \frac{\beta_j}{\sigma_{e_j}^2} (\overline{R}_j - R_f) - a \sum_{h=1}^{H} (g_h / \Psi_h) \sum_{i \in N_h} \frac{b_{i,h}}{\sigma_{e_i}^2} (\overline{R}_i - R_f) \tag{9.9}$$

$$C_h = (1/\Psi_h) \left[\sum_{i \in n_h} \frac{b_{i,h}}{\sigma_{e_i}^2} (\overline{R}_i - R_f) - g_h C_0 \right] \tag{9.10}$$

Reformulating the model by letting R_m reflect the return on an actual portfolio, an "index portfolio" or a "market portfolio" and, hence, a portfolio with zero residual risk, leads to the following covariance matrix:

$$\boldsymbol{\Omega} = \boldsymbol{D} + \boldsymbol{B}\boldsymbol{\Sigma}\boldsymbol{B}^{\boldsymbol{t}} \tag{9.11}$$

where

$$\boldsymbol{D} \equiv \operatorname{diag}\left(0, \sigma^2_{e_1}, \sigma^2_{e_2}, \ldots, \sigma^2_{e_N}\right) \tag{9.12}$$

$$\boldsymbol{\Sigma} \equiv \operatorname{diag}[\sigma^2_m, \sigma^2_{I_1}, \sigma^2_{I_2}] \tag{9.13}$$

$$\boldsymbol{B} \equiv \begin{bmatrix} 1 & 0 & 0 \\ \beta_1 & b_{1,1} & 0 \\ \beta_2 & b_{2,1} & 0 \\ \vdots & \vdots & \vdots \\ \beta_{n_1} & b_{n_1,1} & 0 \\ \beta_{n_1+1} & 0 & b_{n_1+1,2} \\ \beta_{n_1+2} & 0 & b_{n_1+n_2,2} \\ \vdots & \vdots & \vdots \\ \beta_N & 0 & b_{N1,2} \end{bmatrix} \tag{9.14}$$

Let Z_0 denote the position in the "market portfolio" and let $\boldsymbol{Z} = (Z_1, Z_2, \ldots, Z_N)$ be the portfolio positions in the other assets. The first order conditions in the unrestricted case:

$$\boldsymbol{\Omega} \begin{pmatrix} Z_0 \\ \boldsymbol{Z} \end{pmatrix} = \overline{\boldsymbol{R}} - R_f \mathbf{1} \tag{9.15}$$

can be manipulated by the following row operations: Subtract β_j times row 0 from row j for every $j = \{1, 2, \ldots, N\}$. This leaves the reformulated first order conditions as

$$\left(\begin{bmatrix} 0 & \mathbf{0^t} \\ \mathbf{0} & \boldsymbol{D_1} \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & \boldsymbol{b_1} & 0 \\ 0 & 0 & \boldsymbol{b_2} \end{bmatrix} \operatorname{diag}[\sigma^2_m, \sigma^2_{I_1}, \sigma^2_{I_2}] \begin{bmatrix} 1 & \boldsymbol{\beta^t} \\ 0 & \boldsymbol{b_1^t} \quad 0 \\ 0 & 0 \quad \boldsymbol{b_2^t} \end{bmatrix} \right) \begin{pmatrix} Z_0 \\ \boldsymbol{Z} \end{pmatrix} =$$

$$\begin{pmatrix} R_m - R_f \\ R_1 - R_f - \beta_1(R_m - R_f) \\ R_2 - R_f - \beta_2(R_m - R_f) \\ \vdots \\ R_N - R_f - \beta_N(R_m - R_f) \end{pmatrix} \tag{9.16}$$

The first equation simply determines Z_0 residually as

$$Z_0 + \sum_{j=1}^{N} \beta_j Z_j = R_m - R_f \quad \Leftrightarrow \quad Z_0 = R_m - R_f - \sum_{j=1}^{N} \beta_j Z_j \tag{9.17}$$

The rest of these conditions are identical in structure to the multi-group model, in this particular case with two groups. The portfolio positions Z_j are determined as "hedged positions", where the R_m component is hedged out.

Provided that an optimal constrained solution has a non-negative Z_0-component the ranking procedure from the multi-group model can be used to improve the convergence of the LP-algorithm. This requirement is also necessary for the method discussed in Elton, Gruber, and Padberg (1979) to be valid. However, no recipe is given for how to handle cases where it is not fulfilled, and no convergence proof is given.

References

Beasley, J., T.-J. Chang, M. N., and Y. Sharaiha (2000): "Heuristics for cardinality constrained portfolio optimisation," *Computers&Operations Research*, 27:1271–1302.

Elton, E. J. and M. J. Gruber (1995): *Modern Portfolio Theory and Investment Analysis,* Wiley series in Finance. John Wiley & Sons, Inc., Singapore, Singapore, 5 edition.

Elton, E. J., M. J. Gruber, and M. W. Padberg (1976): "Simple criteria for optimal portfolio selection," *The Journal of Finance*, 11:1341–1357.

——— (1977): "Simple rules for optimal portfolio selection: The multi group case," *Journal of Financial and Quantitative Analysis*, 12:329–345.

——— (1978a): "Simple criteria for optimal portfolio selection: Tracing out the efficient frontier," *The Journal of Finance*, 13:296–302.

——— (1978b): "Simple criteria for optimal portfolio selection with upper bounds," *Operations Research.*

——— (1979): "Simple rules for optimal portfolio selection: The multi-index case," in Elton, E. J. and M. J. Gruber, editors, *Portfolio Theory 25 Years Later.* North-Holland, Amsterdam.

Franklin, J. (1980): *Methods of Mathematical Economics,* Springer-Verlag.

Gruber, M. J. (1997): "Personal communication."

Huang, C.-f. and R. H. Litzenberger (1988): *Foundations for Financial Economics,* North-Holland, New York, USA.

Kwan, C. (1984): "Portfolio analysis using single index, multi-index, and constant correlation models: A unified treatment," *The Journal of Finance*, 39(5):1469–1484.

Merton, R. C. (1972): "An analytical derivation of the efficient portfolio frontier," *Journal of Financial and Quantitative Analysis*, 7:1851–1872.

Roll, R. (1977): "A critique of the asset pricing theory's tests," *Journal of Financial Economics*, 4:129–176.

Table 4

	X_0	Z_1	Z_2	Z_3	Z_4	Z_5	Z_6	M_1	M_2	M_3	M_4	M_5	M_6	X_1	X_2	X_3	X_4	X_5	X_6	rhs
X_0	1	19/6	19/6	19/6	19/6	41/15	41/15	−1	−1	−1	−1	−1	−1	0	0	0	0	0	0	85/2
X_1	0	1	1/2	1/2	1/2	1/3	1/3	−1	0	0	0	0	0	1	0	0	0	0	0	10
X_2	0	1/2	1	1/2	1/2	1/3	1/3	0	−1	0	0	0	0	0	1	0	0	0	0	7
X_3	0	1/2	1/2	1	1/2	1/3	1/3	0	0	−1	0	0	0	0	0	1	0	0	0	7
X_4	0	1/2	1/2	1/2	1	1/3	1/3	0	0	0	−1	0	0	0	0	0	1	0	0	6
X_5	0	1/3	1/3	1/3	1/3	1	2/5	0	0	0	0	−1	0	0	0	0	0	1	0	8
X_6	0	1/3	1/3	1/3	1/3	2/5	1	0	0	0	0	0	−1	0	0	0	0	0	1	9/2
X_0	1	0	19/12	19/12	19/12	151/90	151/90	13/6	−1	−1	−1	−1	−1	−19/6	0	0	0	0	0	65/6
Z_1	0	1	1/2	1/2	1/2	1/3	1/3	−1	0	0	0	0	0	1	0	0	0	0	0	10
X_2	0	0	3/4	1/4	1/4	1/6	1/6	1/2	−1	0	0	0	0	−1/2	1	0	0	0	0	2
X_3	0	0	1/4	3/4	1/4	1/6	1/6	1/2	0	−1	0	0	0	−1/2	0	1	0	0	0	2
X_4	0	0	1/4	1/4	3/4	1/6	1/6	1/2	0	0	−1	0	0	−1/2	0	0	1	0	0	1
X_5	0	0	1/6	1/6	1/6	8/9	13/45	1/3	0	0	0	−1	0	−1/3	0	0	0	1	0	14/3
X_6	0	0	1/6	1/6	1/6	13/45	8/9	1/3	0	0	0	0	−1	−1/3	0	0	0	0	1	7/6
X_0	1	0	0	19/18	19/18	179/135	179/135	10/9	10/9	−1	−1	−1	−1	−19/9	−19/9	0	0	0	0	119/18
Z_1	0	1	0	1/3	1/3	2/9	2/9	−4/3	2/3	0	0	0	0	4/3	−2/3	0	0	0	0	26/3
Z_2	0	0	1	1/3	1/3	2/9	2/9	2/3	−4/3	0	0	0	0	−2/3	4/3	0	0	0	0	8/3
X_3	0	0	0	2/3	1/6	1/9	1/9	1/3	1/3	−1	0	0	0	−1/3	−1/3	1	0	0	0	4/3
X_4	0	0	0	1/6	2/3	1/9	1/9	1/3	1/3	0	−1	0	0	−1/3	−1/3	0	1	0	0	1/3
X_5	0	0	0	1/9	1/9	23/27	34/135	2/9	2/9	0	0	−1	0	−2/9	−2/9	0	0	1	0	38/9
X_6	0	0	0	1/9	1/9	34/135	23/27	2/9	2/9	0	0	0	−1	−2/9	−2/9	0	0	0	1	13/18
X_0	1	0	0	0	1/6	16/15	16/15	1/3	1/3	1/3	0	−1	−1	−4/3	−4/3	−4/3	−1	0	0	9/2
Z_1	0	1	0	0	1/4	1/6	1/6	−3/2	1/2	1/2	0	0	0	3/2	−1/2	−1/2	0	0	0	8
Z_2	0	0	1	0	1/4	1/6	1/6	1/2	−3/2	1/2	0	0	0	−1/2	3/2	−1/2	0	0	0	2
Z_3	0	0	0	1	1/4	1/6	1/6	1/2	1/2	−3/2	0	0	0	−1/2	−1/2	3/2	0	0	0	2
M_4	0	0	0	0	−5/8	−1/12	−1/12	−1/4	−1/4	−1/4	1	0	0	1/4	1/4	1/4	−1	0	0	0
X_5	0	0	0	0	1/12	5/6	7/30	1/6	1/6	1/6	0	−1	0	−1/6	−1/6	−1/6	0	1	0	4
X_6	0	0	0	0	1/12	7/30	5/6	1/6	1/6	1/6	0	0	−1	−1/6	−1/6	−1/6	0	0	1	1/2
X_0	1	0	0	0	0	0	0	0	0	0	0	0	0	−1	−1	−1	−1	−1	−1	0
Z_1	0	1	0	0	0	0	0	0	7/15	7/15	0	1/5	0	47/30	−7/15	−7/15	0	−1/5	0	36/5
Z_2	0	0	1	0	7/30	0	3/25	−47/30	7/15	−23/15	7/15	0	1/5	0	−7/15	23/15	−7/15	0	−1/5	6/5
Z_3	0	0	0	1	7/30	0	3/25	7/15	7/15	−23/15	7/15	0	1/5	0	−7/15	−7/15	23/15	0	−1/5	6/5
M_4	0	0	0	0	−37/60	0	−3/50	−7/30	−7/30	−7/30	1	−1/10	0	7/30	7/30	7/30	−1	1/10	0	2/5
Z_5	0	0	0	0	1/10	1	7/25	1/5	1/5	1/5	0	−6/5	0	−1/5	−1/5	−1/5	0	6/5	0	24/5
M_6	0	0	0	0	−3/50	0	−96/125	−3/25	−3/25	−3/25	0	−7/25	1	3/25	3/25	3/25	0	7/25	−1	31/50