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REGIME-SWITCHING STOCK RETURNS AND MEAN REVERSION

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by

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Abstract

We estimate a well-specified two-state regime-switching model for Danish stock returns. The model identifies two regimes which have low return-low volatility and high return-high volatility, respectively. The low return-low volatility regime dominated, except in a few, short episodes, until the beginning of the 70s whereas the 80s and 90s have been characterized by high return and high volatility. We propose an alternative test of mean reversion which allows for multiple regimes with potentially different constant and autoregressive terms and different volatility. Using this test procedure we find mean reversion at 10% but not at 5% significance level which is weaker evidence than produced by estimating a standard autoregressive model for returns. Furthermore, when analyzing contributions of the two regimes we find that the indication of mean reversion is due to the recent high return-high volatility regime only.

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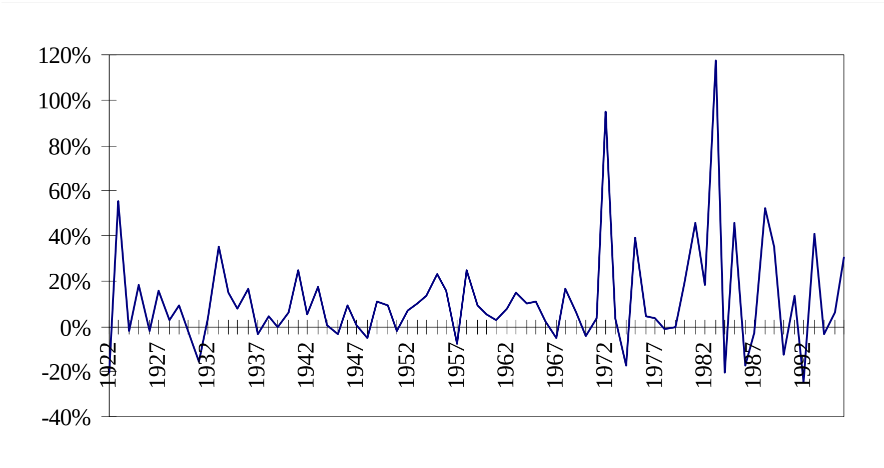
1. Introduction

A plot of Danish stock returns over time suggests that returns were low and relatively stable from the 1920s until the beginning of the 1970s whereas the period since then has been characterized by higher average return and more volatility:

Figure 1. Annual Nominal Stock Returns in Denmark 1922-96



Note: Market portfolio of stocks listed at the Copenhagen Stock Exchange. Data are from the Nielsen, Olesen and Risager (1999) database.

This observation was also made on an informal basis by Nielsen and Risager (1999)[1]. In this paper, we fit a time series model to the nominal return data which allows for the presence of more than one regime. This provides for a formal analysis of whether there have been several regimes and when changes of regime occurred. Furthermore, this approach enables us to test the hypotheses that mean return and volatility are higher in one regime than in the other. Identification of multiple regimes is important for understanding the time series properties of stock returns and may, in particular, be valuable for forecasting purposes.

The plot also indicates that annual stock returns display negative serial correlation (most obviously in the latter part of the sample), ie., that stock prices mean-revert. This question was

[1] However, they view the return in 1972 as an outlier and conclude that the change of regime takes place in 1983.

first raised by Fama and French (1988) and Poterba and Summers (1988) and has been examined for Denmark by Risager (1998). These papers all report weak evidence of mean reversion[2]. The present paper provides an alternative test of this issue within the framework of the regime-switching model. Thus, our approach leads to a mean reversion test which allows for multiple regimes in the return process.

Our procedure takes into account the specific pattern of heteroskedasticity, ie., regime shifts in volatility level, identified by the regime-switching model. There are two related papers by Kim and Nelson (1998) and Kim, Nelson and Startz (1998) in which a similar model for returns is estimated. They standardize returns by estimated volatility and calculate variance ratio and autoregression tests for standardized returns. Our approach, on the other hand, is a parametric test of negative serial correlation which directly utilizes estimates obtained for the regime-switching model.

Furthermore, the paper provides new evidence about the extent to which serial correlation differs across regimes, ie., whether the visual impression, that negative serial correlation is stronger in the latter part of the sample, is correct. In order to apply the tests we calculate analytical expressions for unconditional and state-specific means, variances and serial correlations for the regime-switching model with an autoregressive term.

The following section fits a regime-switching model to our return data. Section 3 derives analytical means and variances of the model and tests hypotheses. Similarly, serial correlation and implications for mean reversion is considered in section 4. Finally, section 5 concludes.

2. Estimating a Regime-Switching Model for Returns

Given the apparent change in behavior of Danish stock returns we are led to estimate a model which accounts for stochastic changes in regime. We employ a two-state version of the model developed by Hamilton (1990). According to this model there is an unobserved state variable, s_t, which takes on the values 0 or 1. The state variable is assumed to follow a Markov chain,

[2] The former paper analyzes real return, the second real and excess return, and the latter real and nominal return. In the present paper, we examine nominal returns.

ie., the transition probabilities satisfy $p_{00} \equiv P(s_t=0|s_{t-1}=0)=P(s_t=0|s_0=i_0,...,s_{t-2}=i_{t-2},s_{t-1}=0)$ and $p_{11} \equiv P(s_t=1|s_{t-1}=1)=P(s_t=1|s_0=i_0,...,s_{t-2}=i_{t-2},s_{t-1}=1)$ for any sequence i_0, ..., i_{t-2} and any t. The observed stock return depends on the state variable:

(1) $$R_t = \mu_0 + (\mu_1 - \mu_0)s_t + \phi_0 R_{t-1} + (\phi_1 - \phi_0)s_t R_{t-1} + \sigma_0 \epsilon_t + (\sigma_1 - \sigma_0)s_t \epsilon_t \ ,$$

where $\epsilon_t \sim$ n.i.d. (0,1).

Thus,

(2) $$R_t|s_t=0 = \mu_0 + \phi_0 R_{t-1} + \sigma_0 \epsilon_t$$

and

(3) $$R_t|s_t=1 = \mu_1 + \phi_1 R_{t-1} + \sigma_1 \epsilon_t \ .$$

Note, that this version of the model allows for distinct μ’s and σ’s, and that an autoregressive term is included in each state.

The parameter vector is estimated by numerically maximizing the log likelihood function. The likelihood function and the maximizing procedure are standard for regime-switching models and described in Hamilton (1994), section 22.4. The algorithm used to evaluate the log likelihood has two other interesting byproducts. First, it is possible to evaluate the probability that a given observation was generated by, say, state 0 conditional on information available at that time (filtered probabilities), ie., current and past stock returns. This provides insight about timing of regime changes. Second, the algorithm generates one-period-ahead probabilities which can be used to construct return forecasts.

Estimating the model described above does not immediately give satisfactory results. The main problem is that the estimate of one of the transition probabilities is at a corner, $\hat{p}_{00}=0$, and that the estimate of the autoregressive term in state 0 is above 1, $\hat{\phi}_0=1.59$. Both of these estimates thus violate the assumptions under which specification tests proposed in Hamilton (1996) are derived. Hence, the distribution of test statistics is unknown. However, informal diagnostic tests of standardized residuals of the three-state model suggests that the three-state model suffers from autocorrelation in the error term, cf. Appendix A. In this formulation, the filtered probabilities conditional on information available at time t only assign three observations to state 0, namely 1923, 1972 and 1983 which all represent years with extraordinary returns (cf. figure 1). Thus, state 0 may be viewed as a state which picks up outliers whereas state 1 is the ordinary state.

To pursue the question of whether there exist two states in addition to the outlier state we estimate a three-state version of the model. This results in an outlier state for 1972 and 1983 and two ordinary states for the remaining observations. The ordinary regimes have low return-low volatility and high return-high volatility, respectively, and the timing of regimes is in line with what we anticipated from looking at data. However, transition probabilities and the autoregressive term of the outlier state cause the same problem as above.

To be able to perform the Hamilton (1996) specification tests of the model and given the indication of misspecification revealed by residual-based tests we therefore choose to introduce dummies for 1972 and 1983 in the two-state model. The two dummy variables have zeroes every year except in 1972 and 1983, respectively, where the value is 1. They are added to equation (1) as two additional variables with potentially distinct coefficients in the two states to allow maximum flexibility. Thus, the resulting model is:

$$(1') \qquad R_t = \mu_0 + (\mu_1 - \mu_0)s_t + \mu_0^{72} d72_t + (\mu_1^{72} - \mu_0^{72})s_t d72_t + \mu_0^{83} d83_t + (\mu_1^{83} - \mu_0^{83})s_t d83_t$$
$$+ \phi_0 R_{t-1} + (\phi_1 - \phi_0)s_t R_{t-1} + \sigma_0 \epsilon_t + (\sigma_1 - \sigma_0)s_t \epsilon_t \ ,$$

where $s_t \in \{0,1\}$ and ϵ_t ~ n.i.d. (0,1). μ_0^{72} and μ_0^{83} are the coefficients to the dummy variables in state 0, and likewise for state 1.[3]

The fundamental difference between the three-state and the dummy model is the assumption of the latter that 1972 and 1983 are abnormal and non-recurring events which can be ignored while fitting a model for the remaining observations. On the other hand, the three-state model views 1972 and 1983 as belonging to a separate, extreme state which there is a (small) positive probability of returning to.

Our choice of the two-state dummy model is motivated by the fact that there are solid economic reasons for treating these years as special. In 1972 Denmark decided to join the EEC and agreed to allow foreign ownership of Danish stocks. In 1983 nominal interest rates were dramatically reduced as a result of the adoption of a fixed exchange rate policy and further capital market liberalizations, and a new pension fund tax was introduced on bond yields only. These events are potential explanations of the outstanding stock returns of these particular years.

The following estimates are obtained for the two-state model with dummies[4]:

[3] The likelihood function is identical to the one presented in Hamilton (1994), p. 692, where the elements in η_t are (using the notation of this paper)

$$\frac{1}{\sqrt{2\pi}\sigma_i}\ \exp\left\{\frac{-(R_t-\mu_i-\mu_i^{72}d72_t-\mu_i^{83}d83_t-\phi_i R_{t-1})^2}{2\sigma_i^2}\right\}\ ,\quad i=0,1$$

[4] Parameter estimates of the two-state dummy model are similar to estimates of the three-state model, cf. Appendix A.

Table 1. Two-state model with dummies for 1972 and 1983, sample 1923-96

μ_0	0.0601 (0.0244)	p_{00}	0.8497 (0.1430)
μ_1	0.1802 (0.0461)	p_{11}	0.8304 (0.1400)
ϕ_0	- 0.0446 (0.0955)	μ_0^{72}	0.8925 (0.0825)
ϕ_1	- 0.3297 (0.1256)	μ_1^{72}	0.7819 (0.5881)
σ_0^2	0.0056 (0.0030)	μ_0^{83}	1.1265 (0.2028)
σ_1^2	0.0385 (0.0126)	μ_1^{83}	1.0582 (0.2150)

Note: Standard errors in parentheses estimated by second derivatives of log likelihood.

Point estimates of μ and σ are smaller in state 0 than in state 1, and as we are going to see in section 3 a non-trivial implication of table 1 is that state 0 is the low return-low volatility state whereas state 1 is characterized by high return and high volatility. ϕ_0 is insignificant but we choose to keep it for use in the next section. Finally, to determine whether the regimes are statistically different we may for example test a hypothesis that the μ's are equal across states. A Wald test rejects this hypothesis (the critical significance level is 0.0244) which confirms that there are 2 distinct regimes.

Note also, that the problem of corner solutions is avoided and that both AR-terms are numerically less than 1. Hence, specification tests suggested by Hamilton (1996) may be applied. These are reported in table 2.

Table 2. Specification tests.

White tests, $\chi^2(4)$		
Autocorrelation	0.7832	(0.9403)
ARCH	6.4781	(0.1677)
Markov property	0.3786	(0.9835)
Lagrange multiplier tests, $\chi^2(1)$		
Autocorrelation in regime 0	0.2250	(0.6394)
Autocorrelation in regime 1	0.0116	(0.9151)
Autocorrelation across regimes	0.5266	(0.4717)
ARCH in regime 0	1.3324	(0.2547)
ARCH in regime 1	0.1414	(0.7079)
ARCH across regimes	0.9079	(0.3454)

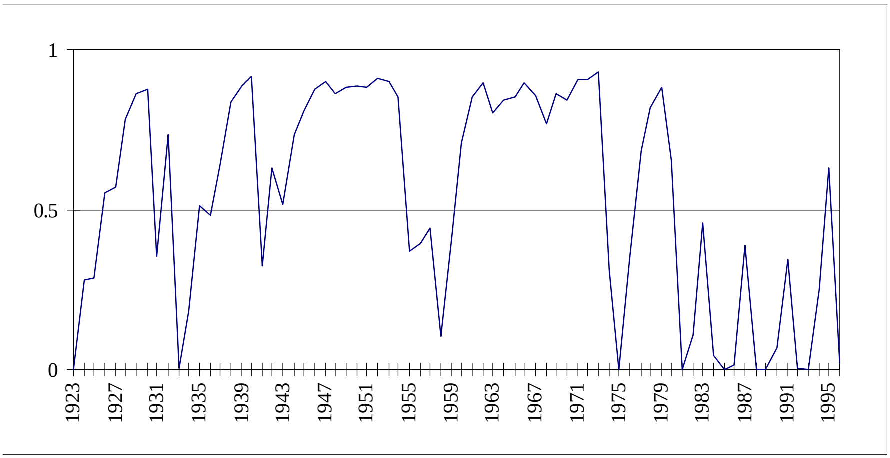
Note: Critical significance levels in parentheses. Large sample tests of Hamilton (1996).

The tests show that the residuals of (1') fulfil the white noise requirements, ie., they are serially uncorrelated and homoskedastic (no ARCH), both within and across regimes. Furthermore, the Markov property of the transition probabilities cannot be rejected, ie., the probabilities of the future state outcome are determined exclusively by the most recent state realization.

The model clearly passes all specification tests at the conventional significance level using large sample distributions. Using the small sample corrections suggested by Hamilton (1996) leads to even clearer acceptance of the model. Furthermore, informal diagnostic tests confirm that standardized residuals is white noise, cf. Appendix B.

We are now ready to analyze the timing of regimes. Figure 2 shows the filtered probabilities, ie., the probability that observation t belongs to state 0 given the information on current and past stock returns available at time t.

Figure 2. Probability that observation t is in state 0 given information available.



This confirms that after a long period of state 0 dominance state 1 has recently become more frequent. Except for a few, short episodes, returns were in the low return-low volatility state with probability greater than one half until 1973. The exceptions are in the beginning of the 20s which was a period of financial distress in Danish financial and industrial companies, the beginning of the 30s which covers both the decline and recovery in the wake of the Wall Street crash, and the latter half of the 50s which marks the beginning of a long business cycle boom. All the episodes occur in periods of volatile stock returns, cf. figure 1. Since 1973, and especially during the 80s and 90s, the high return-high volatility regime has dominated. One possible explanation is that liberalization has made the Danish stock market more vulnerable to foreign volatility.[5] A similar argument is made by Sellin (1996) in relation to a recent Swedish liberalization.

Figure 3 shows the return forecast of the model[6] for time t given information available at t-1.

---

[5] Although the Danish stock market is formally opened to foreigners around 1972, foreign holding of Danish stocks does not accelerate until the beginning of the 1980s, cf. Eskesen et al. (1984). This explanation is consistent with the observation that a persistent regime-shift seems to take place in the beginning of the 1980s.

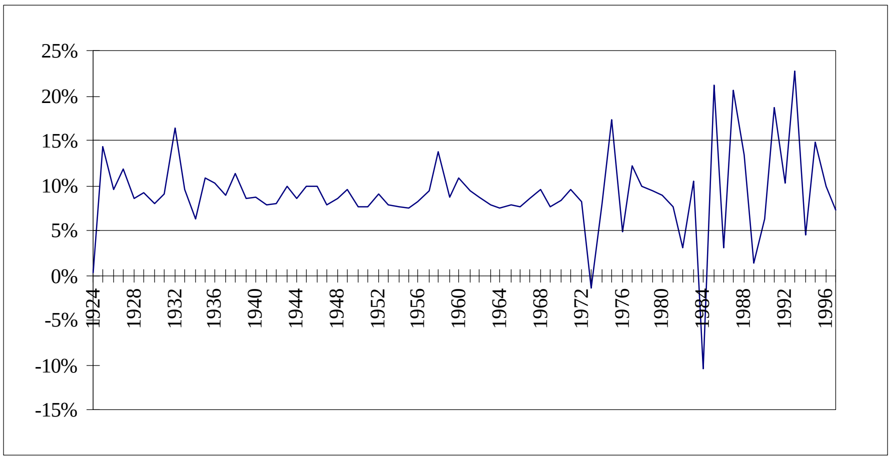
[6] We have excluded the dummy terms in forming forecasts which is natural since the necessity of dummies could not have been anticipated. The forecast is calculated by (note 6 is continued):

$$E(R_t|\Omega_{t-1}) \;=\; P(s_t\!=\!0\,|\Omega_{t-1})(\mu_0 \;+\; \phi_0 R_{t-1}) \;+ P(s_t\!=\!1\,|\Omega_{t-1})(\mu_1 \;+\; \phi_1 R_{t-1})$$

where the probabilities are one-period ahead probabilities, cf. Hamilton (1994), section 22.4.

Assuming that market participants know the return process, we may interpret the model forecast as a measure of market expectations at time t-1 about time t return. We see that the market almost always expected returns within the 5 - 15 per cent per year range in the long period from 1924 to 1972. Since then, and in particular since 1981, market expectations have been extremely volatile and, in fact, more often outside the 5 - 15% range than inside. This reflects that returns have been more volatile in the latter part of the sample and that current returns affect forecasted returns significantly in the state which dominates towards the end.

Figure 3. Model's return forecast at t-1



3. Means and Variances of the Two States

In this section, we calculate means and variances of the return process estimated in the previous section. Both unconditional and conditional means and variances are calculated. We consider an 'ordinary' year, i.e., the dummy terms are ignored.

The calculations in sections 3 and 4 are complicated by the presence of the AR-term and have to our knowledge not been presented elsewhere. It is important to include the AR-term for two reasons. First, table 1 shows that the AR-term is statistically significant. Hence, a model without this component would be misspecified and mean and variance calculations would be invalid. Second, in section 4, we suggest an alternative test for mean reversion in returns which, basically, tests the significance of the AR-term.

3.1. Means

The unconditional mean of model (1) is:

$$
\begin{aligned}
E(R_t) &= P(s_t{=}0)E(R_t|s_t{=}0) + P(s_t{=}1)E(R_t|s_t{=}1) \\
&= \pi_0(\mu_0 + \phi_0 E(R_{t-1}|s_t{=}0)) + \pi_1(\mu_1 + \phi_1 E(R_{t-1}|s_t{=}1)) ,
\end{aligned}
\tag{4}
$$

where $\pi_0 \equiv P(s_t{=}0)=(1-p_{11})/(2-p_{00}-p_{11})$ and $\pi_1 \equiv P(s_t{=}1)=1-\pi_0$ are unconditional (ergodic) probabilities of being in the particular state, cf. Hamilton (1994). Note, that the mean depends on expected return in the previous period conditional on the current state[7]:

$$
\begin{aligned}
E(R_{t-1}|s_t{=}1) &= P(s_{t-1}{=}0|s_t{=}1)E(R_{t-1}|s_{t-1}{=}0) + P(s_{t-1}{=}1|s_t{=}1)E(R_{t-1}|s_{t-1}{=}1) \\
&= pE(R_{t-1}|s_{t-1}{=}0) + (1-p)E(R_{t-1}|s_{t-1}{=}1) ,
\end{aligned}
\tag{5}
$$

where $p \equiv P(s_{t-1}{=}0|s_t{=}1)$ is the probability that the state variable in the previous period was in state 0 given it currently is 1 which can be interpreted as an ‘inverse’ transition probability. Using Bayes’ rule it can be shown that:

$$
\begin{aligned}
p &= \frac{\pi_0 p_{01}}{\pi_0 p_{01} + \pi_1 p_{11}} \\
&= p_{10}
\end{aligned}
\tag{6}
$$

Thus, the inverse transition probability equals the ordinary transition probability.

Assuming covariance stationarity, ie., that means and autocovariances are constant over time, the dating on the right hand side of (5) may be changed:

$$
E(R_{t-1}|s_t{=}1) = pE(R_t|s_t{=}0) + (1-p)E(R_t|s_t{=}1) .
\tag{7}
$$

---

[7] This is derived in Appendix C.

Similarly,

(8) $$E(R_{t-1}|s_t{=}0) \;=\; qE(R_t|s_t{=}0) \;+\; (1-q)E(R_t|s_t{=}1) \;,$$

where $q \equiv P(s_{t-1}{=}0|s_t{=}0)$ is another inverse transition probability. Using Bayes' rule, it can be shown that:

(9) $$q = \frac{\pi_0 p_{00}}{\pi_0 p_{00} + \pi_1 p_{10}} = p_{00}$$

(7) and (8) can be inserted in:

(10) $$E(R_t|s_t{=}0) \;=\; \mu_0 \;+\; \phi_0 E(R_{t-1}|s_t{=}0)$$

(11) $$E(R_t|s_t{=}1) \;=\; \mu_1 \;+\; \phi_1 E(R_{t-1}|s_t{=}1)$$

(derived from (1)) to get two equations in two unknowns. The solutions are[8]:

(12) $$E(R_t|s_t{=}1) \;=\; \frac{A}{B}$$
$$E(R_t|s_t{=}0) \;=\; \frac{\phi_0\;(1-q)\;A \;+\; \mu_0 B}{(1-\phi_0\;q)\;B} \;,$$

where $A \equiv \mu_1 - \phi_0 q \mu_1 + \phi_1 p \mu_0$ and $B \equiv 1 - \phi_0 q - \phi_1 (1-p) - \phi_0 \phi_1 (p-q)$. Finally, insert (12) in (4) to get the unconditional mean.

$E(R_t|s_t{=}i)$ is the expected return in state i. It depends not only on the parameters of state i but also on the parameters of the alternative state. This is due to the AR-terms in returns which force us to consider expected return, and hence the value of the unobserved state variable, in

[8] Assuming $B \neq 0$.

the previous period to form expectations about returns in this period. For example, if $\phi_1>0$ and $p>0$, state 1 expected return increases in μ_0 since there is some probability, p, that the state variable was 0 in the previous period in which case μ_0 affects expected return last period which, in turn, affects expected return in the present period via the positive AR-term in state 1 (ϕ_1).

Given the analytical means in (4) and (12) we are able to estimate:

Table 3. Unconditional and conditional means

$E(R_t)$	0.0955 (0.0177)
$E(R_t\|s_t=0)$	0.0570 (0.0211)
$E(R_t\|s_t=1)$	0.1390 (0.0330)
Wald test, H_0: $E(R_t\|s_t=0)=E(R_t\|s_t=1)$	3.9806 [0.0460]

Note: Each of the means are calculated as a function, $f(\hat{\theta})$, (cf. (4) and (12)) of the estimated parameter vector, $\hat{\theta}=[\hat{\mu}_0, \hat{\mu}_1, \hat{\mu}_0^{72}, \hat{\mu}_1^{72}, \hat{\mu}_0^{83}, \hat{\mu}_1^{83}, \hat{\phi}_0, \hat{\phi}_1, \hat{\sigma}_0^2, \hat{\sigma}_1^2, \hat{p}_{00}, \hat{p}_{11}]'$. Standard errors in parentheses are calculated as: $Std(f(\hat{\theta}))=[\hat{J}'Var(\hat{\theta})\hat{J}]^{1/2}$, where $J=[\partial f/\partial\theta]$. The restriction being tested has been reformulated as $g(\theta)=0$, and the test statistic is calculated as:
$W=g(\hat{\theta})'[\hat{J}'Var(\hat{\theta})\hat{J}]^{-1}g(\hat{\theta})$, where $J=[\partial g/\partial\theta]$. W is asymptotically χ^2 with degrees of freedom equal to number of restrictions (i.e., 1).
Critical significance levels in square brackets.

The estimated unconditional expected return is 9.5% per year which is close to the simple average[9] of 9.1%. State 0 expected return is estimated to 5.7% per year whereas state 1 has an expected return of 13.9%. The Wald test just rejects (at 5% significance) the hypothesis that

[9] From 1923 to 1996 excluding 1972 and 1983.

means are equal in favor of the alternative that means are different.[10] Thus, we are justified in saying that regime 0 has lower expected return than regime 1.

3.2. Variances

Unconditional variance is:

(13) $$Var(R_t) = E(R_t^2) - E(R_t)^2$$

Consider,

(14) $$\begin{aligned} E(R_t^2) &= P(s_t{=}0)E(R_t^2|s_t{=}0) + P(s_t{=}1)E(R_t^2|s_t{=}1) \\ &= \pi_0(\mu_0^2 + 2\mu_0\phi_0 E(R_{t-1}|s_t{=}0) + \phi_0^2 E(R_{t-1}^2|s_t{=}0) + \sigma_0^2) + \\ &\quad \pi_1(\mu_1^2 + 2\mu_1\phi_1 E(R_{t-1}|s_t{=}1) + \phi_1^2 E(R_{t-1}^2|s_t{=}1) + \sigma_1^2) \end{aligned}$$

using the model, that is, (2) and (3).

In this expression, we have that

(15) $$\begin{aligned} E(R_{t-1}^2|s_t{=}0) &= qE(R_{t-1}^2|s_{t-1}{=}0) + (1-q)E(R_{t-1}^2|s_{t-1}{=}1) \\ E(R_{t-1}^2|s_t{=}1) &= pE(R_{t-1}^2|s_{t-1}{=}0) + (1-p)E(R_{t-1}^2|s_{t-1}{=}1) \end{aligned}$$

Assuming covariance-stationarity, we need to solve[11]

(16) $$\begin{aligned} E(R_t^2|s_t{=}0) &= E[(\mu_0 + \phi_0 R_{t-1} + \sigma_0\epsilon_0)(\mu_0 + \phi_0 R_{t-1} + \sigma_0\epsilon_0)|s_t{=}0] \\ &= \mu_0^2 + 2\mu_0\phi_0 E(R_{t-1}|s_t{=}0) + \phi_0^2(qE(R_t^2|s_t{=}0) + (1-q)E(R_t^2|s_t{=}1)) + \sigma_0^2 \end{aligned}$$

and a similar expression for $E(R_t^2|s_t{=}1)$ to obtain $E(R_t^2)$. The solutions for $E(R_t^2|s_t{=}i)$

---

[10] In addition to the Wald test, we have performed a Likelihood Ratio test of the same hypothesis which has a critical significance level of 0.0643 leading to acceptance of the hypothesis at 5%. We have more confidence in the Wald test, however, since filtered probabilities change completely under the restriction which in our opinion makes the test hard to interpret. Possibly, the existence of multiple local maxima of the unrestricted likelihood function reduce the power of Likelihood Ratio tests.

[11] $E(R_{t-1}|s_t{=}0)$ is known from section 3.1.

are in appendix D. Subtracting the squared means derived earlier gives expressions for unconditional and conditional variances.

Unconditional and conditional variances can now be estimated:

Table 4. Unconditional and conditional variances

$Var(R_t)$	0.0246 (0.0074)
$Var(R_t \mid s_t=0)$	0.0056 (0.0030)
$Var(R_t \mid s_t=1)$	0.0425 (0.0143)
Wald test, H_0:$Var(R_t \mid s_t=0)=Var(R_t \mid s_t=1)$	7.7977 [0.0052]

Note: See note to table 3 where 'f' now relates to the variance formulae derived above. Standard errors in parentheses and critical significance levels in square brackets.

The estimated unconditional standard error of annual returns is 15.7% which should be compared to the sample standard error of 16.4%.[12] State 0 standard deviation is 7.5% whereas state 1 standard deviation is 20.6%. The hypothesis that conditional variances are equal is strongly rejected with a critical significance level of less than 1 per cent.[13] Hence, volatility is lower in state 0 than in state 1.

Finally, a Wald test rejects the joint hypothesis that both means and variances are equal across states (the critical significance level is 0.0013).

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[12] From 1923 to 1996 excluding 1972 and 1983.

[13] A Likelihood Ratio test of the hypothesis has a critical significance level of 0.1193 leading to acceptance of H_0. However, the test is not easily interpretable, cf. footnote 9.

4. Serial Correlation: Test for Mean Reversion

The question of whether stock prices are mean-reverting has received a lot of attention since the papers by Fama and French (1988) and Poterba and Summers (1988). A number of studies have produced evidence of mean reversion using variance-ratio and autoregression tests, see Risager (1998) for an analysis of the Danish return data. In this section, we provide evidence based on an alternative test procedure which has the important feature that it explicitly allows for regime-shifts in the return process.

We choose to focus attention on first order serial correlation. Specification tests in table 2 and Appendix B show no sign of autocorrelation in the error term, so any higher order serial correlation is due to first order serial correlation. We calculate the analytical first order serial correlations of the two-state Markov switching model, see appendix E. Then we obtain point estimates and standard errors:

Table 5. Unconditional and conditional first order serial correlation

$Corr(R_t,R_{t-1})$	-0.1993
	(0.1104)
$Corr(R_t,R_{t-1}\|s_t=0)$	0.0297
	(0.2482)
$Corr(R_t,R_{t-1}\|s_t=1)$	-0.3340
	(0.1214)
Wald test, H_0: $Corr(R_t,R_{t-1})=0$	3.2567
	[0.0711]
Wald test, H_0: $Corr(R_t,R_{t-1}\|s_t=0)=0$	0.0143
	[0.9048]
Wald test, H_0: $Corr(R_t,R_{t-1}\|s_t=1)=0$	7.5669
	[0.0059]

Note: See note to table 3 where ‘f’ relates to the serial correlation formulae displayed in Appendix E.

Our estimate of first order serial correlation across regimes is -0.2 which is significantly less than zero at 10% significance level but cannot be rejected to be zero at the 5% level. Hence, there is weak evidence of mean reversion in nominal stock returns which is consistent with findings of others.[14]

Interestingly, the same hypothesis has a critical significance level of 0.0042 in a standard one-regime AR 1-specification with dummies for 1972 and 1983 and using OLS standard errors which leads to clear acceptance of mean reversion.[15] Hence, allowing for multiple regimes results in much less support for mean reversion than the standard AR-regression introduced by Fama and French (1988). This finding is consistent with the results of Kim and Nelson (1998) who also conclude that accounting for the observed pattern of heteroskedasticity stemming from regime-switching volatility of returns weakens the evidence of mean reversion according to autoregression tests.[16]

Thus, it is important to take account of heteroskedasticity when making inference about mean reversion, in particular, since the critical significance levels are close to the conventional significance level even small changes may have large qualitative importance for conclusions. Although OLS gives consistent estimates of coefficients, a procedure which allows for heteroskedasticity (of the correct form) leads to more efficient inference. Moreover, usual OLS estimates of variances including coefficient standard errors are biased. Heteroskedasticity consistent standard errors (such as White) improve inference asymptotically, but may have problems in small samples. For example, in our case, using White standard errors only increases the critical significance level to 0.0064, whereas we found a critical significance level of around 7%.

Our regime-switching model includes the standard one-regime model as a special case, and, hence, is more general. Therefore, we have more confidence in results of the regime-switching model. We interpret the conflicting inference as evidence of weaknesses of the standard

[14] Risager (1998) finds slightly more support for mean reversion in real than in nominal returns which indicates that the critical significance level would be slightly less than 7.11% if our analysis were applied to real returns.

[15] The estimated coefficient to lagged returns is -0.235 with a t-statistic of -2.957. These results are similar to the findings in Nielsen and Risager (1999) and Risager (1998).

[16] A similar conclusion has been found for the variance ratio test by Kim, Nelson and Startz (1998).

approach.

Our analysis highlights two important points. First, in the presence of multiple persistent regimes which have the feature that some but not all regimes exhibit mean reversion, it is important to have observations from each regime in order to draw correct inference. In the case of nominal Danish stock returns it is particularly important to have enough observations after the beginning of the 80s to be able to detect two regimes. This parallels the socalled peso problem encountered in the exchange rate literature, see e.g. Evans (1996), ie., in order to identify a process with rare, discrete events, a large sample is needed.[17] Second, there are two sources to negative serial correlation if the true return generating process is regime-switching. First of all, a negative autoregressive term creates mean reversion as in the usual one-state AR case. But, even if the autoregressive term is zero in both states serial correlation may be different from zero just because the process shifts between states (assuming these have different means).

Within our framework, we are able to distinguish serial correlation of the two states. As table 5 shows, our estimate of serial correlation is only negative in state 1. In fact, only in state 1 is serial correlation significantly different from zero. Hence, we conclude that the weak evidence of mean reversion presented in table 5 is (mainly) a result of serial correlation in the high return-high volatility state which has dominated the most recent decades. This is in contrast to results for the US which indicate that mean reversion was stronger before World War II than after, see Kim, Nelson and Startz (1991) and Kim and Nelson (1998).

The evidence of mean reversion parallels the findings in Risager (1998). Using standard autoregressive and variance ratio tests, he finds weak support of the mean reversion hypothesis. Furthermore, the paper suggests splitting the sample into subsamples. This analysis indicates that mean reversion has been stronger in the most recent part of the sample, that is, since the 1970s. This conclusion is consistent with the results of the regime-switching model in the present paper.

[17] We conjecture that since our model is constructed to identify regime-shifts, it will stand a better chance of solving peso problems and lead to more reliable inference on mean reversion in small samples.

Given the strong presence of mean reversion in recent years, what should we expect for the future? This basically depends on whether one believes that the current regime is absorbing or not. From a purely statistical point of view, there is a probability of returning to the no-mean-reversion state which implies that unconditional serial correlation is the right measure, thus suggesting only weak evidence for mean reversion. From an economic point of view, however, it is essential to focus on the underlying factors which cause regime changes and, in particular, to analyze whether all the variables causing the most recent regime-shift are reversible. It is perhaps not likely that the liberalizations, which we argue led to the latest transition to high volatility, will be reversed within a foreseeable future. However, other factors, such as a decrease of US stock market volatility, may be able to cause a return to low volatility. In other words, we use capital market liberalizations as one (of several) component to explain the latest transition to high volatility but do not view deliberalization as necessary for a return to the low volatility regime. Hence, economic considerations have ambigous implications for the question of mean reversion.

5. Conclusion

We have estimated a well-specified two-state regime-switching model for Danish stock returns. The model identifies two regimes which have low return-low volatility and high return-high volatility, respectively. The low return-low volatility regime dominated, except in a few, short episodes, until the beginning of the 70s whereas the 80s and 90s have been characterized by high return and high volatility.

We propose an alternative test of mean reversion which allows for multiple regimes with potentially different constant and autoregressive terms and different volatility. Using this test procedure we find mean reversion at 10% but not at 5% significance level. This is weaker evidence than produced by the standard method of testing for significance of the AR-term in a one-regime autoregressive model. Furthermore, when analyzing contributions of the two regimes we find that the indication of mean reversion is due to the recent high return-high volatility regime only.

The regime-switching model has also been applied by Kim and Nelson (1998) and Kim, Nelson and Startz (1998) on stock returns using US data. Our approach differs by allowing for an autoregressive term and by incorporating regime-shifts in the mean. Both features are shown to be relevant for Danish data.

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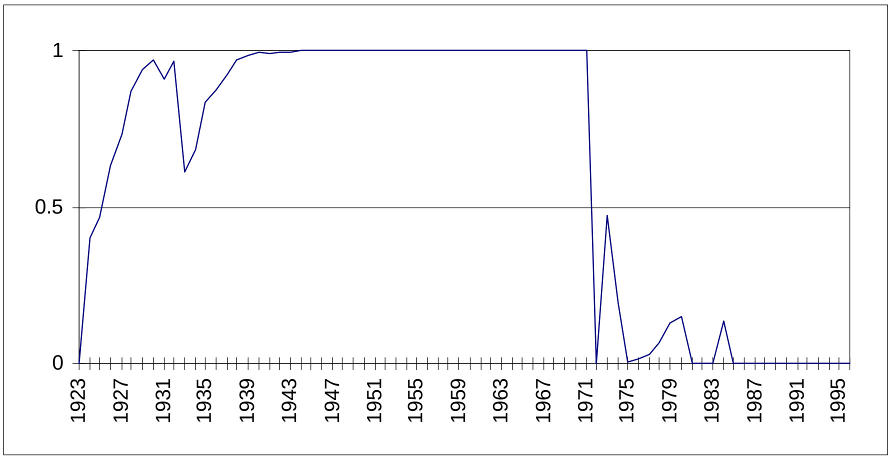
Appendix A: Three-State Model

Parameter estimates

μ_0	0.0781 (0.0167)	p_{00}	0.9703 (0.0228)
μ_1	0.1614 (0.0493)	p_{01}	0.0000
μ_2	0.8923 (0.0021)	p_{02}	0.0297
ϕ_0	- 0.0888 (0.1163)	p_{10}	0.0000
ϕ_1	- 0.2616 (0.1234)	p_{11}	0.9328 (0.0495)
ϕ_2	1.5922 (0.0127)	p_{12}	0.0672
σ_0^2	0.0091 (0.0019)	p_{20}	0.2741
σ_1^2	0.0440 (0.0130)	p_{21}	0.7259
σ_2^2	0.0000 (0.0000)	p_{22}	0.0000

Note: Standard errors in parentheses estimated by second derivatives of log likelihood. Omitted standard errors cannot be calculated due to corner solutions.

Filtered probabilities for state 0



The outlier state has filtered probabilities close to 1 in 1923, 1972 and 1983 and zero otherwise.

All point estimates of the three-state model are within one standard deviation of the two-state dummy model estimates. The main difference is that the regimes are estimated to be more persistent in the three-state model. This has the implication that inference about the state and the timing of regime shifts is much clearer than in the two-state model. Another difference between the models is that the three-state model assigns some probability to the event that s_t returns to the outlier state.

Diagnostic tests of standardized residuals:[18]

	Test statistic	Critical significance level
AR(1)	0.0000	0.9967
AR(2)	3.2539	0.0445 *
AR(3)	2.1393	0.1030
AR(4)	1.853	0.1286
AR(5)	1.5575	0.1839
AR(6)	1.5812	0.1663
AR(7)	1.3459	0.2433
AR(8)	1.2564	0.2818
ARCH(1)	0.0661	0.7978
Normality	0.9729	0.6148

[18] Standardized residuals are calculated as in Appendix B except for the extra state.

Appendix B: Analysis of Standardized Residuals of Two-State Dummy Model

Standardized residuals are calculated as the difference between actual and fitted return divided by conditional standard deviation, ie., the square root of (derived in Nielsen and Olesen, 1999):

$$Var(R_t|\Omega_{t-1}) = P(s_t=0|\Omega_t)\sigma_0^2 + P(s_t=1|\Omega_t)\sigma_1^2 + P(s_t=0|\Omega_t)P(s_t=1|\Omega_t)(E(R_t|\{\Omega_{t-1},s_t=0\}) - E(R_t|\{\Omega_{t-1},s_t=1\}))^2 \ ,$$

where Ω_t contains information about current and past stock returns. Fitted returns are:

$$\hat{R}_t = P(s_t=0|\Omega_T)(\mu_0 + \mu_0^{72}d72_t + \mu_0^{83}d83_t + \phi_0 R_{t-1}) + P(s_t=1|\Omega_T)(\mu_1 + \mu_1^{72}d72_t + \mu_1^{83}d83_t + \phi_1 R_{t-1})$$

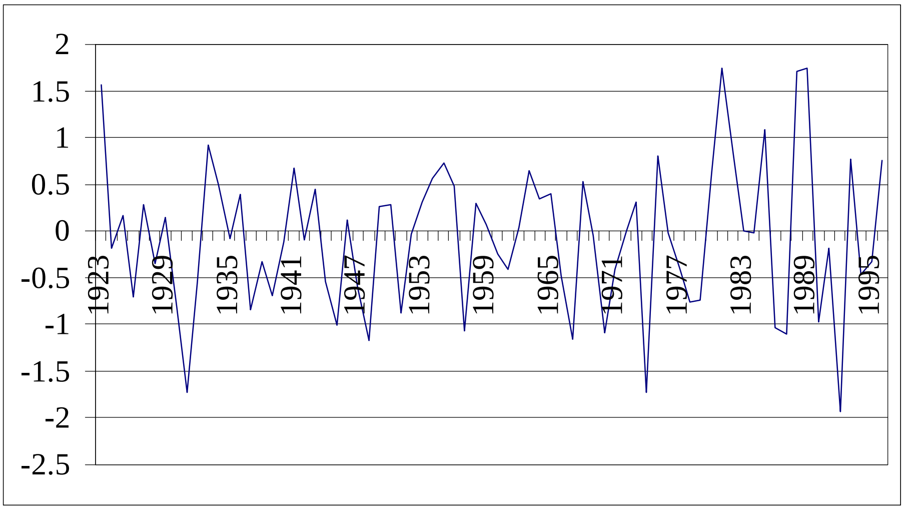
which is conditioned on information on past stock returns and uses filtered probabilities for each state (that is, probabilities conditioned on Ω_T which includes all available stock returns of the sample). The standardized residuals are estimates of ϵ_t in (1').

The standardized residuals have been tested for autocorrelation from lag 1 to 8, ARCH and normality:

	Test statistic	Critical significance level
AR(1)	0.0390	0.8440
AR(2)	1.5828	0.2126
AR(3)	1.2332	0.3042
AR(4)	0.9278	0.4530
AR(5)	0.8111	0.5458
AR(6)	0.7359	0.6225
AR(7)	0.6219	0.7360
AR(8)	0.6334	0.7468
ARCH(1)	1.6207	0.2071
Normality	0.4076	0.8156

The following plot confirms that the standardized residuals are well-behaved:

Standardized Residuals



Appendix C: Derivation of (5)

$$
\begin{aligned}
E(R_{t-1} \mid s_t = 1) &= \int R_{t-1} f(R_{t-1} \mid s_t = 1) dR_{t-1} \\
&= \int R_{t-1} \sum_{j=0}^{1} f(R_{t-1}, s_{t-1} = j \mid s_t = 1) dR_{t-1} \\
&= \int R_{t-1} \sum_{j=0}^{1} f(R_{t-1} \mid s_{t-1} = j, s_t = 1) P(s_{t-1} = j \mid s_t = 1) dR_{t-1} \\
&= \int R_{t-1} \sum_{j=0}^{1} f(R_{t-1} \mid s_{t-1} = j) P(s_{t-1} = j \mid s_t = 1) dR_{t-1} \\
&= \sum_{j=0}^{1} \int R_{t-1} f(R_{t-1} \mid s_{t-1} = j) P(s_{t-1} = j \mid s_t = 1) dR_{t-1} \\
&= \sum_{j=0}^{1} P(s_{t-1} = j \mid s_t = 1) \int R_{t-1} f(R_{t-1} \mid s_{t-1} = j) dR_{t-1} \\
&= \sum_{j=0}^{1} P(s_{t-1} = j \mid s_t = 1) E(R_{t-1} \mid s_{t-1} = j) \\
&= P(s_{t-1} = 0 \mid s_t = 1) E(R_{t-1} \mid s_{t-1} = 0) + P(s_{t-1} = 0 \mid s_t = 1) E(R_{t-1} \mid s_{t-1} = 0)
\end{aligned}
$$

Appendix D: Solutions for $E(R_t^2|s_t=i)$

$$E(R_t^2|s_t=0) = \frac{CD + \phi_0^2(1-q)[(1-\phi_0^2 q)E + \phi_1^2 pD]}{(1-\phi_0^2 q)C}$$

$$E(R_t^2|s_t=1) = \frac{(1-\phi_0^2 q)E + \phi_1^2 pD}{C} ,$$

where

$$C = 1-\phi_0^2 q-\phi_0^2\phi_1^2 p(1-q)-\phi_1^2(1-p)+\phi_0^2\phi_1^2(1-p)q$$

$$D = \mu_0^2+2\mu_0\phi_0 E(R_{t-1}|s_t=0)+\sigma_0^2$$

$$E = \mu_1^2+2\mu_1\phi_1 E(R_{t-1}|s_t=1)+\sigma_1^2$$

Appendix E: Serial Correlation

Unconditional first order serial correlation is defined as (assuming covariance stationarity):

$$Corr(R_t,R_{t-1}) = \frac{Covar(R_t,R_{t-1})}{Var(R_t)}$$

Thus, we need:

$$\begin{aligned} E(R_tR_{t-1}) = {} & \pi_0(q[\mu_0^2+\mu_0\phi_0E(R_{t-1}|s_t{=}0)+\mu_0\phi_0E(R_t|s_t{=}0)]+ \\ & (1-q)[\mu_0\mu_1+\mu_0\phi_1E(R_{t-1}|s_t{=}1)+\mu_1\phi_0E(R_t|s_t{=}1)])+ \\ & \pi_1(p[\mu_0\mu_1+\mu_1\phi_0E(R_{t-1}|s_t{=}0)+\mu_0\phi_1E(R_t|s_t{=}0)]+ \\ & (1-p)[\mu_1^2+\mu_1\phi_1E(R_{t-1}|s_t{=}1)+\mu_1\phi_1E(R_t|s_t{=}1)])+ \\ & (\pi_0q\phi_0+\pi_1p\phi_1)(\phi_0E(R_tR_{t-1}|s_t{=}0)+\sigma_0^2)+ \\ & (\pi_0(1-q)\phi_0+\pi_1(1-p)\phi_1)(\phi_1E(R_tR_{t-1}|s_t{=}1)+\sigma_1^2) \end{aligned}$$

Hence, we must solve

$$\begin{aligned} E(R_tR_{t-1}|s_t{=}0) = {} & q(\mu_0^2+\mu_0\phi_0E(R_{t-1}|s_t{=}0)+\mu_0\phi_0E(R_t|s_t{=}0)+\phi_0^2E(R_tR_{t-1}|s_t{=}0)+\phi_0\sigma_0^2)+ \\ & (1-q)(\mu_0\mu_1+\mu_0\phi_1E(R_{t-1}|s_t{=}1)+\mu_1\phi_0E(R_t|s_t{=}1)+\phi_0\phi_1E(R_tR_{t-1}|s_t{=}1)+\phi_0\sigma_1^2) \end{aligned}$$

and a similar expression for $E(R_tR_{t-1}|s_t{=}1)$. The solutions are:

$$E(R_tR_{t-1}|s_t{=}0) = \frac{CF + (1-q)\phi_0\phi_1[(1-\phi_0^2q)G + p\phi_0\phi_1F]}{(1-\phi_0^2q)C}$$

$$E(R_tR_{t-1}|s_t{=}1) = \frac{(1-\phi_0^2q)G + p\phi_0\phi_1F}{C},$$

where

$$\begin{aligned} F = {} & q\mu_0^2+(1-q)\mu_0\mu_1+q\mu_0\phi_0E(R_{t-1}|s_t{=}0)+(1-q)\mu_0\phi_1E(R_{t-1}|s_t{=}1)+q\mu_0\phi_0E(R_t|s_t{=}0)+ \\ & (1-q)\mu_1\phi_0E(R_t|s_t{=}1)+q\phi_0\sigma_0^2+(1-q)\phi_0\sigma_1^2 \\ G = {} & p\mu_0\mu_1+(1-p)\mu_1^2+p\mu_1\phi_0E(R_{t-1}|s_t{=}0)+(1-p)\mu_1\phi_1E(R_{t-1}|s_t{=}1)+p\mu_0\phi_1E(R_t|s_t{=}0)+ \\ & (1-p)\mu_1\phi_1E(R_t|s_t{=}1)+p\phi_1\sigma_0^2+(1-p)\phi_1\sigma_1^2 \end{aligned}$$

Inserting these solutions and the results from the previous sections above gives $E(R_t R_{t-1})$. Subtract $E(R_t)^2$ to obtain $Covar(R_t, R_{t-1})$. Similarly for state dependent covariances.