



**Copenhagen
Business School**
HANDLSHØJSKOLEN

Department of Economics

Copenhagen Business School

Working paper 15-2005

A THREE-PERIOD SAMUELSON-DIAMOND GROWTH MODEL

Niels Blomgren-Hansen

A Three-period Samuelson-Diamond Growth Model

Niels Blomgren-Hansen
Department of Economics, Copenhagen Business School

November 2005

Abstract

Samuelson (1958) analyses a three-period model, whereas Diamond (1965) considers a two-period model. This difference poses the question whether the insights derived by analysing the simple two-period model carry over in the more complicated three-period case. They do. The Samuelson model (no productive capital) has only one positive solution ($r = n$); however, this root is unstable. The Diamond model (no non-productive abode of purchasing power) has also only one positive solution; the root is stable but inefficient. In a model with both productive capital and a non-productive abode of purchasing power, the inefficient Diamond solution becomes unstable and the socially optimal solution becomes stable.

1 Household and firm behavior

Assume that each person lives not just two but three periods. He works in two periods and lives from his saving for yet another period.

As in the two-period case (Niels Blomgren-Hansen 2005), a person belonging to generation t maximizes a simple log-linear utility function

$$U(t) = \ln c_{1t} + \frac{1}{1 + \rho} \cdot \ln c_{2t} + \frac{1}{(1 + \rho)^2} \cdot \ln c_{3t}$$

subject to his budget constraint,

$$c_{1t} + \frac{1}{1 + r_{t+1}} \cdot c_{2t} + \frac{1}{(1 + r_{t+1}) \cdot (1 + r_{t+2})} \cdot c_{3t} \leq w_t + \frac{1}{1 + r_{t+1}} \cdot w_{t+1}$$

The f.o.c. is readily derived

$$\begin{aligned} c_{1t} &= \left(w_t + \frac{1}{1+r_{t+1}} \cdot w_{t+1} \right) \cdot \left(\frac{(1+\rho)^2}{1+(1+\rho)+(1+\rho)^2} \right) \\ c_{2t} &= \left(w_t + \frac{1}{1+r_{t+1}} \cdot w_{t+1} \right) \cdot \left(\frac{(1+\rho) \cdot (1+r_{t+1})}{1+(1+\rho)+(1+\rho)^2} \right) \\ c_{3t} &= \left(w_t + \frac{1}{1+r_{t+1}} \cdot w_{t+1} \right) \cdot \left(\frac{(1+r_{t+1}) \cdot (1+r_{t+2})}{1+(1+\rho)+(1+\rho)^2} \right) \end{aligned}$$

The firms' problem is unaffected of the number of periods. They maximize their profit per worker with respect to the capital stock per worker in each period,

$$\pi_t = y_t - k_t \cdot r_t$$

Assuming a C-D production

$$y_t = k_t^\alpha$$

the f.o.c. may be expressed in each of the following forms

$$\begin{aligned} r_t &= \alpha \cdot k_t^{\alpha-1}; & k_t &= \left(\frac{\alpha}{r_t} \right)^{\frac{1}{1-\alpha}}; & k_t &= \alpha \cdot \frac{y_t}{r_t} \\ w_t &= (1-\alpha) \cdot k_t^\alpha = (1-\alpha) \cdot \left(\frac{\alpha}{r_t} \right)^{\frac{\alpha}{1-\alpha}} \end{aligned}$$

2 Diamond solution

The size of the population 'born' in period t is normalized to 1. The population increases by a constant rate of growth, n .

Total production in period t is

$$Y_t = y_t \cdot \left(1 + \frac{1}{1+n} \right) = k_t^\alpha \cdot \left(1 + \frac{1}{1+n} \right) = \left(\frac{\alpha}{r_t} \right)^{\frac{\alpha}{1-\alpha}} \cdot \left(1 + \frac{1}{1+n} \right)$$

The crucial assumption made by Diamond (1965) is that the stock of productive capital in period $t+1$ is equal to the wealth (accumulated purchasing power) at the end of period t

$$K_{t+1} = W_t$$

$$\begin{aligned}
K_{t+1} &= k_{t+1} \cdot (1 + (1+n)) = \left(\frac{\alpha}{r_{t+1}} \right)^{\frac{1}{1-\alpha}} \cdot (1 + (1+n)) \\
W_t &= s_{1t} + s_{2t-1} \cdot \frac{1+r_t}{1+n} + s_{1t-1} \cdot \frac{(1+r_t) \cdot (1+r_{t-1})}{(1+n)^2} \\
&= \left(w_t - \left(w_t + \frac{1}{1+r_{t+1}} \cdot w_{t+1} \right) \cdot \left(\frac{(1+\rho)^2}{1+(1+\rho)+(1+\rho)^2} \right) \right) \\
&\quad + \left(w_t - \left(w_{t-1} + \frac{1}{1+r_t} \cdot w_t \right) \cdot \left(\frac{(1+\rho) \cdot (1+r_t)}{1+(1+\rho)+(1+\rho)^2} \right) \right) \cdot \frac{1}{1+n} \\
&\quad + \left(w_{t-1} - \left(w_{t-1} + \frac{1}{1+r_t} \cdot w_t \right) \cdot \left(\frac{(1+\rho)^2}{1+(1+\rho)+(1+\rho)^2} \right) \right) \\
&\quad \cdot \frac{(1+r_t)}{(1+n)}
\end{aligned}$$

or - by collecting terms - as

$$\begin{aligned}
k_{t+1} \cdot (1 + (1+n)) &= -w_{t+1} \cdot \frac{1}{1+r_{t+1}} \cdot \left(\frac{(1+\rho)^2}{1+(1+\rho)+(1+\rho)^2} \right) \\
&\quad + w_t \cdot \left(\frac{1 - \left(\frac{(1+\rho)^2}{1+(1+\rho)+(1+\rho)^2} \right) + \frac{1}{1+n} \left(1 - \left(\frac{(1+\rho)+(1+\rho)^2}{1+(1+\rho)+(1+\rho)^2} \right) \right)}{1} \right) \\
&\quad + w_{t-1} \cdot \frac{(1+r_t)}{(1+n)} \cdot \left(1 - \left(\frac{(1+\rho) + (1+\rho)^2}{1+(1+\rho)+(1+\rho)^2} \right) \right)
\end{aligned}$$

In steady state the stocks of (desired) capital and wealth as fractions of total production and as functions of r reduce to

$$\begin{aligned}
K/Y &= \frac{\alpha}{r} \cdot (1+n) \\
W/Y &= \left(\frac{(1-\alpha)}{1 + \frac{1}{1+n}} \right) \cdot \left[\left(\frac{(1+\rho)^2}{1+r} + (1+\rho)^2 + \frac{(2+r)}{1+n} \cdot ((1+\rho) + (1+\rho)^2) \right) \right]
\end{aligned}$$

The two functions are sketched in figure A1.

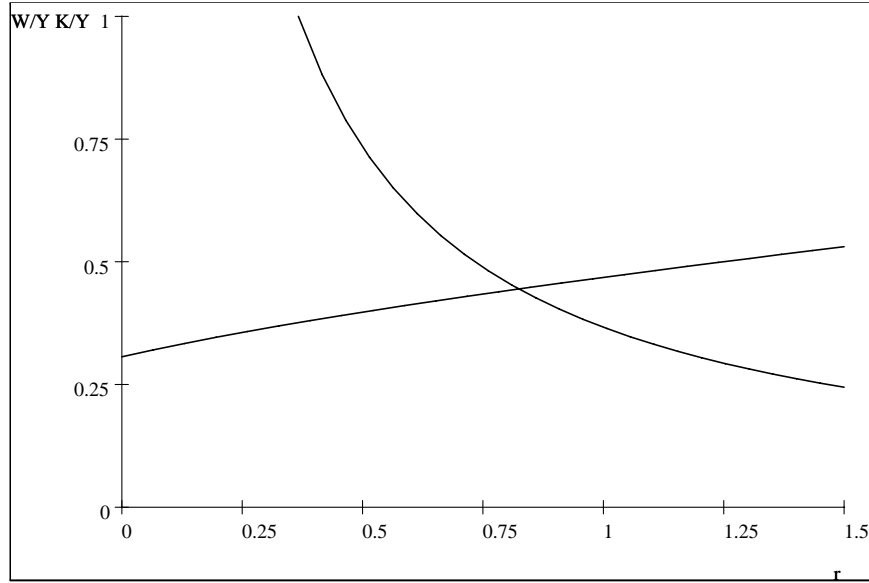


Figure A1: Wealth and capital as functions of the rate of interest (for $\alpha = 0.3333$; $n = 0.10$; $\rho = 0.05$)

The three-period Diamond model has three roots. For $\alpha = 0.3333$; $n = 0.10$; $\rho = 0.05$, the solution is $r = -0.843, -5.237, 0.825$. Only the positive root, $r = 0.825$, makes economic sense (as a negative rate of interest implies a negative stock of capital).

3 Ordinary macro-economic equilibrium solution

Total consumption is the consumption of the three generations living in period t .

$$\begin{aligned}
 C_t &= c_{1t} + c_{2t-1} \cdot \frac{1}{1+n} + c_{3t-2} \cdot \frac{1}{(1+n)^2} \\
 &= \left(w_t + \frac{1}{1+r_{t+1}} \cdot w_{t+1} \right) \cdot \left(\frac{(1+\rho)^2}{1+(1+\rho)+(1+\rho)^2} \right) \\
 &\quad + \left(w_{t-1} + \frac{1}{1+r_t} \cdot w_t \right) \cdot \left(\frac{(1+\rho) \cdot (1+r_t)}{1+(1+\rho)+(1+\rho)^2} \right) \cdot \frac{1}{1+n} \\
 &\quad + \left(w_{t-2} + \frac{1}{1+r_{t-1}} \cdot w_{t-1} \right) \cdot \left(\frac{(1+r_{t-1}) \cdot (1+r_t)}{1+(1+\rho)+(1+\rho)^2} \right) \cdot \frac{1}{(1+n)^2}
 \end{aligned}$$

which may be written as

$$\begin{aligned}
C_t = & w_{t-2} \cdot \left(\frac{(1+r_{t-1}) \cdot (1+r_t)}{(1+n)^2} \right) \cdot \frac{1}{1+(1+\rho)+(1+\rho)^2} \\
& + w_{t-1} \cdot \frac{(1+r_t)}{(1+n)} \cdot \left(\frac{\left(\frac{1}{1+(1+\rho)+(1+\rho)^2} \right) \cdot \frac{1}{1+n}}{\left(\frac{(1+\rho)}{1+(1+\rho)+(1+\rho)^2} \right)} \right) \\
& + w_t \cdot \left(\frac{\left(\frac{(1+\rho)}{1+(1+\rho)+(1+\rho)^2} \right) \cdot \frac{1}{1+n} + \left(\frac{(1+\rho)^2}{1+(1+\rho)+(1+\rho)^2} \right)}{\left(\frac{(1+\rho)^2}{1+(1+\rho)+(1+\rho)^2} \right)} \right) \\
& + w_{t+1} \cdot \left(\frac{(1+\rho)^2}{1+(1+\rho)+(1+\rho)^2} \cdot \frac{1}{1+r_{t+1}} \right)
\end{aligned}$$

In steady state this expression reduces to

$$C_t = \frac{(1-\alpha) \cdot y}{1+(1+\rho)+(1+\rho)^2} \cdot \left(\frac{\left(\frac{(1+r)^2}{(1+n)^2} + \frac{(1+r)}{(1+n)} \cdot \frac{1}{1+n} \right) + \left(\frac{(1+r)}{(1+n)} + \frac{1}{1+n} \right) \cdot (1+\rho) + \left(1 + \frac{1}{1+r} \right) \cdot (1+\rho)^2}{\left(\frac{(1+\rho)^2}{1+(1+\rho)+(1+\rho)^2} \right)} \right)$$

As we assume instant adjustment and perfect foresight Investments is the increase in the desired stock of capital from period t to period $t+1$,

$$I_t = k_{t+1} \cdot ((1+n) + 1) - k_t \cdot \left(1 + \frac{1}{1+n} \right)$$

which in steady state reduces to

$$I_t = k \cdot \left(1 + \frac{1}{1+n} \right) \cdot n = \alpha \cdot \frac{y}{r} \cdot n \cdot \left(1 + \frac{1}{1+n} \right)$$

From this exercise it becomes obvious that $r = n$ is a solution, as

$$C_t + I_t = (1-\alpha) \cdot y \cdot \left(1 + \frac{1}{1+n} \right) + \alpha \cdot y \cdot \left(1 + \frac{1}{1+n} \right) = y \cdot \left(1 + \frac{1}{1+n} \right) = Y_t$$

However, there may be more solutions, and the question of the stability of the roots remains.

In figure A2 $S/Y = 1 - C/Y$ and I/Y are sketched as functions of r (for technical reasons with r plotted along the abscissa).

$$\begin{aligned}
S/Y &= \left[\frac{1 - \frac{(1-\alpha)}{1+(1+\rho)+(1+\rho)^2} \cdot \frac{1}{(1+\frac{1}{1+n})}}{\left(\frac{\left(\frac{(1+r)^2}{(1+n)^2} + \frac{(1+r)}{(1+n)} \cdot \frac{1}{1+n} \right) + \left(\frac{(1+r)}{(1+n)} + \frac{1}{1+n} \right) \cdot (1+\rho) \right) + \left(1 + \frac{1}{1+r} \right) \cdot (1+\rho)^2} \right] \\
I/Y &= \alpha \cdot \frac{n}{r}
\end{aligned}$$

Contrary to the two-period model, S/Y is here not a linear function of r .

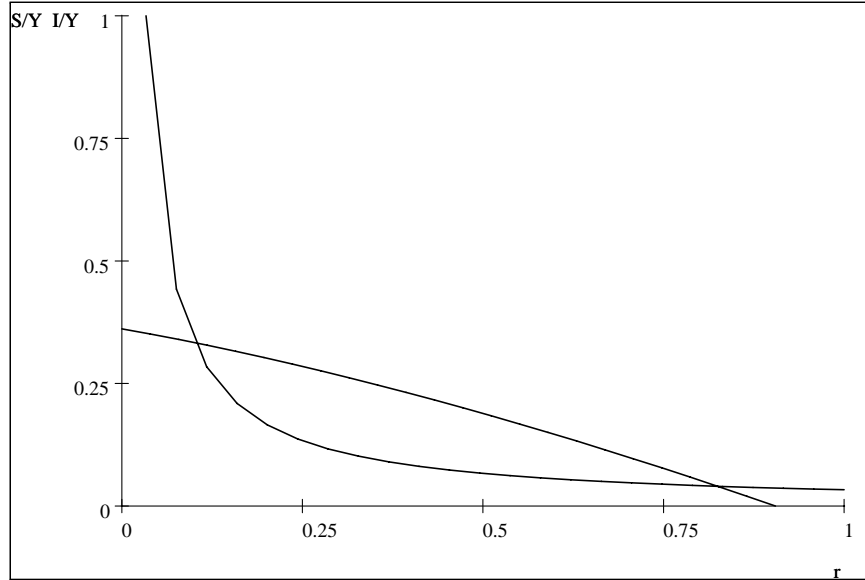


Figure A2: Saving and investment of functions of the rate of interest (for $\alpha = 0.3333$; $n = 0.10$; $\rho = 0.05$)

Expressed in terms of the rate of interest, the model has four roots: -5.234 , -0.843 , 0.1 , 0.825 . The three roots are identical with the roots derived above when closing the model as Diamond proposes. The fourth root is the socially optimal root, $r = n = 0.1$. Again, only the two positive roots make economic sense.

As the graph illustrates, $r = n$ is a stable root. In case of $r > n = 0.1$, saving (supply of loanable funds) exceeds investment (demand for loanable funds); the rate of interest is competed down. If, on the contrary, the rate of interest exceeds the higher root, $r = 0.825$, the demand for loanable funds exceeds the supply of loanable funds; the rate of interest rises further.

4 The Samuelson case

Samuelson (1958) analyses a consumption-loan model with no real capital (and, consequently, $\alpha = 0$). In this setting the rate of interest, r , may be negative. However, the model makes only economic sense if the present value of saving is positive, i.e if $r \geq -1$. The model has three roots. For $n = 0.10$ and $\rho = 0.05$,

the equilibrium condition reduces to

$$S/Y = 0.22496 - 0.13732r^2 - \frac{0.18319}{r+1} - 0.57056r = 0$$

the solution of which is $r = -4.593; -0.662; 0.10$.

The model is sketched in figures A3(i) and A3(ii) below

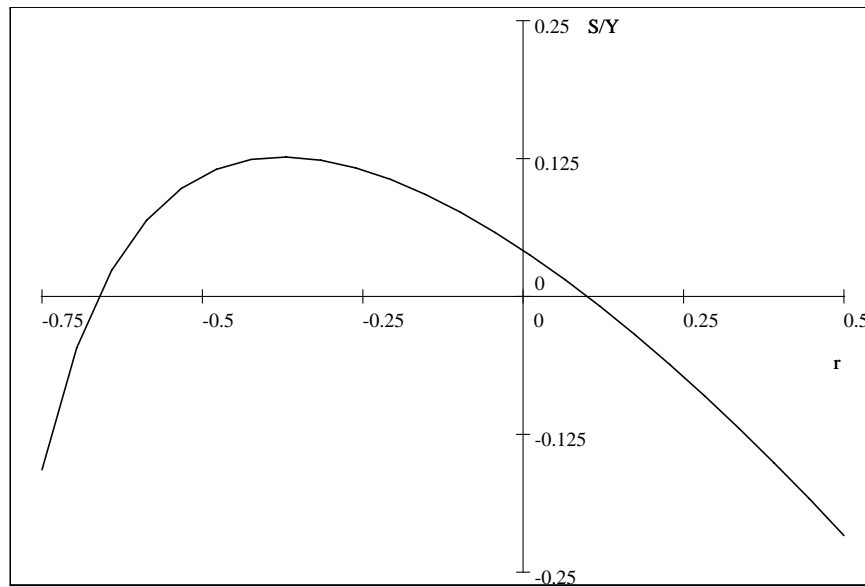


Figure A3(i): The Samuelson model for $n = 0.10$ and $\rho = 0.05$ in the range $r > -1$

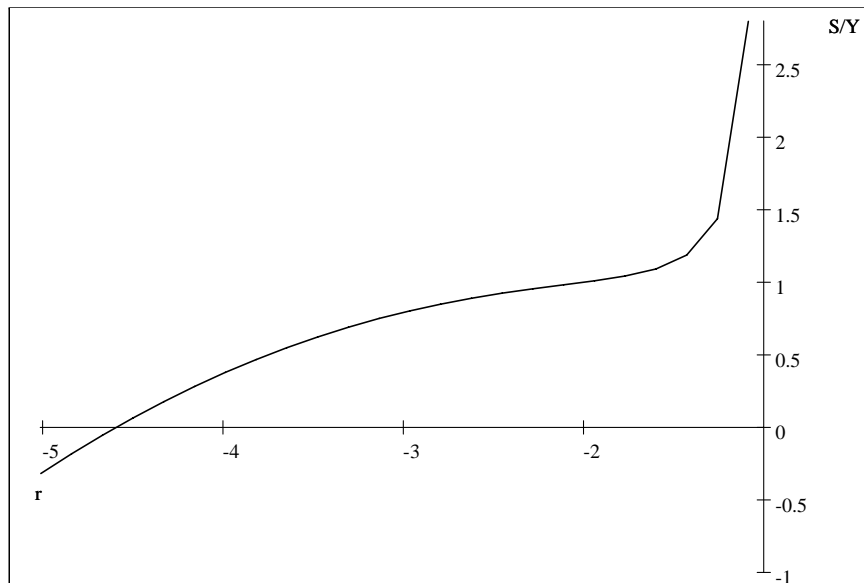


Figure A3(i): The Samuelson model for $n = 0.10$ and $\rho = 0.05$ in the range $r < -1$

The graphs confirm Samuelson's conclusion: The socially optimal solution, $r = n$, is *unstable* in a pure consumption loan model. At $r = n = 0.1$, the saving function cuts the zero-line from above. Saving is a negative function of the rate of interest. At a rate higher than n the model indicates excess demand for loanable funds and upward pressure on the rate of interest. On the contrary, both negative roots are stable.

The crucial difference between the Samuelson model and our model is the fact that we allow for the existence of productive capital and not just 'chocolate papers' (Samuelson's metaphor). Allowing for productive capital and assuming a production function that implies that investments are more elastic with respect to the rate of interest than saving at the socially optimal rate of interest, makes this rate of interest stable.

The crucial difference between our model and the Diamond model is the fact that we allow for 'chocolate papers' and consequently, do not impose the unrealistic assumption that wealth (in the sense of stored purchasing power) must equal the value of the stock of physical productive capital.

References

Diamond, P.A. (1965), National debt in a Neoclassical Growth Model. *AER*, vol.55.

Blomgren-Hansen, N. (2005), A Note on the (In)stability of the Diamond Growth Model (WP, Department of Economics, CBS)

Phelps, E.S. (1961), The Golden Rule of Accumulation: A Fable for Growthmen. *AER*, vol.51.

Samuelson, P.A. (1958), An Exact Consumption-Loan Model of Interest With and Without the Social Contrivance of Money. *JPE*, vol LXVI