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TAXATION AND SYSTEMATIC RISK
UNDER DECREASING RETURNS TO SCALE

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Abstract

Lund (2002a) showed in a CAPM-type model how tax depreciation schedules affect required expected returns after taxes. Even without leverage higher tax rates implied lower betas when tax deductions were risk free. Here they are risky, and marginal investment is taxed together with inframarginal in an analytical model of decreasing returns. With imperfect loss offset tax claims are analogous to call options. The beta of equity is still decreasing in the tax rate, but increasing in the underlying volatility. The results are important if market data are used to infer required expected returns, and in discussions of tax design.

KEYWORDS: Corporate tax, depreciation, imperfect loss offset, decreasing returns, cost of capital, uncertainty

JEL CLASSIFICATION NUMBERS: F23, G31, H25

1 Introduction

This paper presents an analytical model of imperfect loss offset, values tax claims based on capital market equilibrium relationships, and explores the consequences for the after-tax systematic risk of equity. In order for a marginal investment to be taxed together with infra-marginal investments, an analytical production function with decreasing returns to scale is introduced. In order to be consistent with capital market equilibrium, it is assumed that the firm pays a fee to obtain this technology.

The public economics literature on the relationship between taxation and the cost of capital typically neglects uncertainty,[1] while the finance literature typically focuses on the tax treatment of debt or on differential taxation of different forms of capital income on the investors' hands.[2] The points made in the present paper are valid (also) when there is no debt, and irrespective of differential taxation of the shareholders.

Different corporate tax systems split the risk between the firm and the government in different ways. There is not one general required expected rate of return after corporate tax. An average will not do either. Instead one must consider how each tax system affects the risk characteristics of the after-corporate-tax cash flow. In particular, depreciation schedules, interest deductibility, and loss offset are important.

Lund (2002a) showed how a standard corporate income tax affects the systematic risk of after-tax cash flows when investments are not expensed, but tax deductible according to a depreciation schedule. That study first considered the case of full (perfect) loss offset (or more generally, a tax position known with full certainty). Perfect loss offset will be a good approximation for some large firms,[3] but in reality future tax deductions connected with the marginal investment[4] are not completely risk free. As an alternative Lund (2002a) considered a case where the marginal investment was supposed to yield operating income in the second period which would be taxed alone. The depreciation deduction would be lost to the extent that it exceeded the second-period operating income.

More realistically, the marginal investment will be taxed together with other activities in the firm. In the present paper there are supposed to be decreasing returns to scale, while all uncertainty originates from a single stochastic variable, the second-period output price.

This is just a first step towards more realism, but sufficiently complicated for a separate study. Further extensions may be a multi-period model and/or more stochastic variables, such as prices of more than one output, not perfectly correlated.

In order to get results consistent with an equilibrium in capital markets, the firm will have a net market value of zero. In the CAPM jargon,[5] the firm is on the security market line. The profitable DRS (decreasing returns to scale) technology can only be obtained by paying a "fee" for it, e.g., by buying a license or patent, or by doing research and development. The magnitude of this expense is determined competitively. It is supposed to be immediately tax deductible, although alternative tax treatments are considered in the appendix. Apart from DRS the assumptions will be as in Lund (2002a). Only parts of the motivation are repeated here.

It is possible to arrive at results on the effect of taxation under uncertainty which are powerful, consistent with the assumptions of the present paper, but derived in a more straightforward way. Fane (1987) is an example, relying on value additivity, considering separately the risk of each of the cash flow's elements.[6] The method in the present paper is more suitable not only for imperfect loss offset, but generally when the systematic risk of the net after-tax cash flow of a firm is of interest. This is consistent with the practice of most firms,[7] and is essential whenever data from financial markets are used to estimate required expected rates of return for new projects.[8]

Levy and Arditti (1973) observe that taxes with depreciation schedules affect the required expected rate of return after tax. Their model is an extension of Modigliani and Miller (1963), introducing depreciable assets in their model, but maintaining their assumptions of perpetual projects and full loss offset. Lund (2003) discusses their model and claims that a more realistic alternative turns their results around.[9] The appendix below shows the exact relationship between the results of the present paper and those of Lund (2003).

Galai (1988) (very briefly, p. 81) and Derrig (1994) both discuss the effect of a corporate income tax on the systematic risk of equity based on the CAPM. Derrig does not observe the necessity of solving for the expected rate of return of an after-tax marginal project.[10]

Both Levy and Arditti (1973) and Derrig (1994) consider only one simple tax system, and assume that the firm is certain to be in tax position. The present paper (like Lund

(2002a)) is an extension in both respects. Galai (1988) considers both risky debt and a risky tax position, but only one tax system. None of the previous studies consider DRS.

The paper is a supplement to the empirical work on estimating marginal tax rates of firms, taking tax carry-forward and carry-back into consideration. Some central references are Shevlin (1990), Graham (1996), and Shanker (2000). While the empirical studies are more realistic by taking multi-period effects into account, the present model gives analytical solutions, identifying which factors are likely to have important effects.

Section 2 presents a two-period model in which the firm produces with decreasing returns to scale and pays taxes with certainty. Section 3 introduces uncertainty about whether taxes are paid. While these two sections focus on the after-tax cost of capital, section 4 gives results on the cost of capital before taxes. Section 5 contains additional discussion of some aspects of the model. Section 6 concludes. Some proofs and additional details are in the appendix.

2 The model when the tax position is certain

A firm invests in period 0 and produces in period 1, only. The firm considers an investment project with decreasing returns to scale. It is free to choose the scale of investment. The optimal choice is endogenous, determined by the tax system and other parameters in each case below. In this way we also characterize the minimum required expected return to equity in each case.

Assumption 1: *The firm maximizes its market value according to a tax-adjusted Capital Asset Pricing Model,*

$$E(r_i) = r\theta + \beta_i[E(r_m) - r\theta], \tag{1}$$

where $r > 0$ and $\theta \in (0, 1]$.[11]

This allows for differences in the tax treatment on the hands of the firm's owners of income from equity and income from riskless bonds, reflected in the tax parameter θ.[12] In a discussion of taxation and the CAPM it seems reasonable to allow for $\theta < 1$, but it has no consequences for the results which follow. The standard CAPM with $\theta = 1$ is all that is needed.

When various tax systems are considered below, these are assumed not to affect the capital market equilibrium. This will be a good approximation if they apply in small sectors of the economy (e.g., natural resource extraction), or abroad in economies ("host countries") which are small in relation to the domestic one.[13] This is thus a partial equilibrium analysis.

The ("home") economy where the firm's shares are traded may have a tax system, which is exogenously given in the analysis, and reflected in θ. This is often referred to as a "personal" tax system, even though the owners may be firms or other institutions.

A consequence of the CAPM is that the claim to any uncertain cash flow X, to be received in period 1, has a period-0 value of

$$\varphi(X) = \frac{1}{1+r\theta}[E(X) - \lambda_\theta \operatorname{cov}(X, r_m)], \tag{2}$$

where $\lambda_\theta = [E(r_m) - r\theta]/\operatorname{var}(r_m)$. Equation (2) defines a valuation function φ to be applied below.

A product price, P, will most likely not have an expected rate of price increase which satisfies the CAPM.[14] A claim on one unit of the product will satisfy the CAPM, however, so that the beta value of P should be defined in relation to the return $P/\varphi(P)$,

$$\beta_P = \frac{\operatorname{cov}(\frac{P}{\varphi(P)}, r_m)}{\operatorname{var}(r_m)}. \tag{3}$$

It is possible to express this more explicitly, without the detour via $\varphi(P)$, namely as

$$\beta_P = \frac{1+r\theta}{E(P)\frac{\operatorname{var}(r_m)}{\operatorname{cov}(P,r_m)} - [E(r_m) - r\theta]}, \tag{4}$$

cf. equation (4) of Ehrhardt and Daves (2000).

Assumption 2: *In period 0 the firm invests an amount $I > 0$ in a project. In period 1 the project produces a quantity $Q = f(I) > 0$ to be sold at an uncertain price P. The production function f has $f' > 0, f'' < 0$. The joint probability distribution of (P, r_m) is exogenous to the firm, and $\operatorname{cov}(P, r_m) > 0$. There is no production flexibility; Q is fixed after the project has been initiated.*

The assumption of $\operatorname{cov}(P, r_m) > 0$ can easily be relaxed. It is only a convenience in order to simplify the verbal discussions below.

In order to arrive at some of the results in what follows, the specification

$$f(I) = \omega I^{\alpha}, \tag{5}$$

is used in some places, with $\omega > 0$ and $\alpha \in (0,1)$, so that $\alpha = f'(I)I/f(I)$, a constant elasticity.

Assumption 3: *The fee which is paid for the right to undertake the investment project is M_0. This is competitively determined among firms with the same leverage (see Assumption 4) and tax position (see Assumption 5), so that the net value after taxes to the firm of paying this fee, borrowing, undertaking the project in optimal scale, and paying taxes, is zero. The sequence of events in period 0 is as follows: (a) The authorities determine the tax system for both periods. (b) The firm pays the fee M_0 (and possibly borrows some fraction of this, see Assumption 4). (c) The firm determines how much to invest, I (and possibly borrows some fraction of this, see Assumption 4).*

Assumption 4: *A fraction $(1-\eta) \in [0,1)$ of the financing need in period 0 is borrowed. This fraction is independent of the investment decision and of the tax system. The loan B is repaid with interest with full certainty in period 1.*

The financing need is equal to $M_0 + I$ minus the immediate tax relief for these costs, if any.[15]

Debt is introduced only because of the prominence of debt in the traditional literature on taxes and the cost of capital. In the present paper the results can be derived with zero debt. The assumptions of independence between financing and investment, and between (after-tax) financing and taxes, are those underlying the simplest standard derivation of the WACC, and therefore the appropriate set of assumptions here.[16]

It should be kept in mind that Assumption 4 concerns the formal project-related borrowing by the firm. When applied to the subsidiary of a multinational, this may be tax motivated and differ from the net project-related borrowing undertaken by the multinational and its subsidiaries taken together. The lender to one subsidiary is often another subsidiary of the same multinational. Also, the possibility of transfer pricing is neglected here.

The assumption of default-free debt is a common simplification, and should be acceptable for the purposes of this paper. Although the firm's operating revenue may turn out too low to repay the debt, it is realistic in many cases that the loan is effectively guaranteed by a parent company. Galai (1988) focuses on risky debt in a similar model.

It will be clear below that there may be tax advantages to debt. When the firm decides on the optimal investment I, the fee M_0 is already paid, and a fraction of this is borrowed. These are given magnitudes when the investment decision is made, so this decision is independent of the tax advantage of the M_0-related borrowing.

Assumption 5: *A tax at rate* $t \in [0, 1)$ *will be paid with certainty in the production period. The tax base is operating revenue less* $(grB + cI)$*. There is also a tax relief of* $t(M_0 + aI)$ *in period 0. The constants* $g, c,$ *and* a *are in the interval* $[0, 1]$*; moreover,* $t[a + c/(1 + r)] < 1$.

This general formulation allows for accelerated depreciation with, e.g., $a > 0$ and $a + c = 1$, or a standard depreciation interpreted (since there is only one period with production) as $a = 0, c = 1$. There is usually full interest deduction, i.e., $g = 1$, but the Brown (1948) cash flow tax has $g = 0$, and some transfer pricing regulations might require $0 < g < 1$. The requirement $t[a + c/(1 + r)] < 1$ precludes "gold plating incentives," i.e., the tax system carrying more, in present value terms, than one hundred percent of an investment cost.[17]

Assumption 5 implies that a negative tax base gives a negative tax. While this is unrealistic for most tax systems when the project stands alone, it is not at all unrealistic when the marginal project is added to other activity which is more profitable. An alternative assumption for the second period is considered in section 3. For the first period, however, no alternative is considered. This could rely on an assumption that firms only start projects in periods in which they are in tax position to benefit immediately from deductions allowed in the first period. This does not explain how most firms get started in the first place.

Since M_0 is immediately tax deductible with full certainty, it is really the after-tax payment $M = M_0(1 - t)$ which is of interest to us. From Assumption 3 it follows that it is this payment, plus any tax advantage connected with a related borrowing, which is adjusted competitively so that the total net value is zero. It is also M, not M_0, which determines the financing need, so that borrowing is a fraction of M. If the authorities change the tax rate, the relation between M_0 and M changes, but it is still M which is determined competitively, and thus M appears in the equations to follow, except towards the end of the appendix, in which a more general tax system is considered.

2.1 Case F: No borrowing, $\theta = 1$

This first case (case F for risk Free deductions) is considered to demonstrate as simply as possible the method used in Lund (2002a), applied both to the problem solved there and to the case with decreasing returns to scale. This will show the distinction between two concepts, marginal beta and average beta, which are important in what follows. Consider the case with $\eta = 1$ (no borrowing), and $\theta = 1$ (no tax discrimination effect in the capital market where the firm's stock is traded).

In the case FC (C for CRS) of a marginal project alone, considered in Lund (2002a), the cash flow to equity in period 1 is

$$X_{FC} = PQ(1-t) + tcI, \tag{6}$$

where Q in the CRS model replaces $f(I)$ in the DRS model. For each set of tax and other parameters, Q is set so that the project is exactly marginal. The market value of a claim to this is

$$\varphi(X_{FC}) = \varphi(P)Q(1-t) + \frac{tcI}{1+r}. \tag{7}$$

For a marginal project the expression must be equal to the financing need after borrowing and taxes, $I(1-ta)$, so that

$$I(1-ta) = \varphi(X_{FC}) = \varphi(P)Q(1-t) + \frac{tcI}{1+r}, \tag{8}$$

which implies

$$\frac{\varphi(P)Q}{I} = \frac{1 - ta - \frac{tc}{1+r}}{1-t}. \tag{9}$$

The beta value of equity is a value-weighted average of the beta values of the elements of the cash flow. From (6) this is

$$\beta_{FC} = \frac{\varphi(P)Q(1-t)}{\varphi(X_{FC})}\beta_P = \frac{1 - ta - \frac{tc}{1+r}}{1-ta}\beta_P, \tag{10}$$

cf. Lund (2002a), eq. (9) and (12). The main conclusion in that paper is that due to the tax depreciation schedule, the beta of equity is decreasing in the tax rate under a corporate income tax. Under a pure cash flow tax there is no such effect of the tax rate.

The intuition behind the tax effect is as follows: A pure cash flow tax ($a = 1, c = 0$) does not affect the beta of equity, since it is equivalent, cash-flow-wise, to the government assuming the role of a shareholder. As compared with a cash flow tax, the typical corporate income tax postpones some deductions in the form of a tax depreciation schedule, and these will be less risky at the margin than the future operating income. Risk-wise this postponement is similar to a loan from the firm to the authorities, and thus it has the opposite effect of leverage: It reduces the systematic risk of equity. Since the result rests critically on the risk characteristics of the tax value of depreciation deductions, the focus of the present paper is on making more realistic assumptions about the uncertainty of the firm's tax position.

One main assumption in this and what follows is that the underlying systematic risk, β_P, is unaffected by changes in the tax system. This corresponds to the partial-equilibrium assumption made in relation to the capital market, cf. Assumption 1. The assumption is more realistic the smaller the investment project is, and the smaller the coverage of this tax system is, in relation to the market for the output.

Consider now the DRS case, FD. Instead of technically adjusting Q to find the characteristics of a marginal project, as above, there is now a first-order condition which determines I, and one can then solve for the fee which makes the overall addition to net value equal to zero.

The cash flow to equity in period 1 is

$$X_{FD} = Pf(I)(1-t) + tcI, \tag{11}$$

The market value of a claim to this is

$$\varphi(X_{FD}) = \varphi(P)f(I)(1-t) + \frac{tcI}{1+r}. \tag{12}$$

The firm chooses the optimal scale in order to maximize $\pi_{1D}(I) = \varphi(X_{FD}) - I(1-ta)$. The first-order condition for a maximum is

$$\varphi(P)f'(I) = \frac{1 - ta - \frac{tc}{1+r}}{1-t}. \tag{13}$$

The fee will be set so that $M = \pi_{1D}(I) = \varphi(X_{FD}) - I(1 - ta)$, which implies that the total payment in period 0 is

$$M + I(1 - ta) = \varphi(X_{FD}). \tag{14}$$

The beta value of equity is a value-weighted average of the beta values of the elements of the cash flow. From (11) this is

$$\beta_{FD} = \frac{\varphi(P)f(I)(1 - t)}{\varphi(X_{FD})}\beta_P. \tag{15}$$

It is helpful here to introduce the parameterized production function, (5). Together with the first order condition (13), this gives

$$\beta_{FD} = \frac{1 - ta - \frac{tc}{1+r}}{1 - ta - \frac{tc}{1+r}(1 - \alpha)}\beta_P, \tag{16}$$

which again is decreasing in the tax rate as long as $c > 0$. As α approaches unity (i.e., CRS), β_{FD} approaches β_{FC} (and equilibrium M approaches zero).

Observe that $\beta_{FC} < \beta_{FD}$ when $tc(1 - \alpha) > 0$. The two different expressions for the beta of equity will be called *marginal beta* and *average beta*, respectively. They are both relevant as descriptions of systematic risk within the same project. The average beta will describe the systematic risk of the project as a whole, and in particular, the systematic risk of the shares in a firm with only this project. The marginal beta is still the relevant one for decision making at the margin, which may be decentralized within the firm. After the cost M_0 has been sunk, the correct beta for calculating the required expected rate of return is the marginal beta.

The expressions for the two betas will be somewhat more complicated in the cases which follow, in particular when the tax position is uncertain. But the difference will reappear. So far we can observe that the origin of the difference is the tax depreciation schedule. Only if $tc > 0$, will the difference depend on α. The costs M_0 and I are treated differently by the tax system. Since M_0 is immediately deductible, the tax system is partly a cash flow system, partly based on a depreciation schedule. This will be realistic for many forms of "fees." Immediate deduction is usual for licenses and patents, but also for the firm's own R&D.

Apart from this difference, there is no fundamental distinction between M_0 and I from the firm's point of view. They are both paid in the same period, and the output next period, Q, could have been written as a function of their sum. (In order to represent the same underlying reality, this function would have the value zero for total costs less than M_0.) Another way to see the same point is that both betas are equal to β_P if the tax rate is zero, but also if the tax rate is positive while c is zero. In the present model with a sunk cost and DRS thereafter, it is the difference in tax treatment of these costs which creates the two different betas of equity. This will also be the case below when the tax position is uncertain.

When capital budgeting is presented in standard textbooks, this distinction between marginal and average beta is not mentioned. There may be good reasons for this: There are many details of projects and tax systems which have to be left out in a textbook. Lund (2002a, 2003) emphasizes the importance of tax systems for after-tax required expected rates of return. If firms continue to rely mainly on one such required rate for their net after-tax cash flow,[18] they should be aware of tax effects, and not only on the value of debt, which has been the traditional focus. If they want to infer the requirement from capital market data for their own shares, they should be aware that these data (if the model is true) reflect average beta, not marginal beta. In addition to the need to "unlever" and "untax" betas, there is now a need to "unaverage" betas.[19]

2.2 Case B: Allow for borrowing and $\theta < 1$

We now allow $\eta \in (0,1]$ and $\theta \in (0,1]$. (Case B for Borrowing.) The marginal beta is derived in equations (9) and (12) in Lund (2002a), and can be written (in a form which will be easily comparable with results to follow) as

$$\beta_{BC} = \frac{(1-ta)\Lambda - \frac{ct}{1+r\theta}}{(1-ta)\Lambda} \cdot \frac{\Lambda}{\eta}\beta_P, \tag{17}$$

where

$$\Lambda \equiv \eta + (1-\eta)\frac{1+r(1-tg)}{1+r\theta} > 0 \tag{18}$$

gives the relative tax savings from leverage.[20]

Consider now case BD, the DRS version of case B. Let the total borrowing, B, be the sum of two elements, $B_M = (1-\eta)M$ and $B_I = (1-\eta)I(1-ta)$.[21] The total cash flow to equity in period 1 will be

$$X_{BD} = Pf(I) - t(Pf(I) - cI - grB) - (1+r)B, \tag{19}$$

with market value in period 0 equal to

$$\varphi(X_{BD}) = \varphi(P)f(I)(1-t) + \frac{tcI}{1+r\theta} - \frac{(1+r(1-tg))B}{1+r\theta}. \tag{20}$$

After the fee is sunk cost, however, the relevant loan is B_I, not B. The cash flow is

$$X_{BDI} = Pf(I) - t(Pf(I) - cI - grB_I) - (1+r)B_I, \tag{21}$$

with market value in period 0 equal to

$$\varphi(X_{BDI}) = \varphi(P)f(I)(1-t) + \frac{tcI}{1+r\theta} - \frac{(1+r(1-tg))B_I}{1+r\theta}, \tag{22}$$

which can be rewritten as

$$\varphi(X_{BDI}) = \varphi(P)f(I)(1-t) + \frac{I}{1+r\theta}[tc - (1+r(1-tg))(1-\eta)(1-ta)]. \tag{23}$$

The net value of the project exclusive of the fee is

$$\pi_{BD}(I) = \varphi(X_{BDI}) - I(1-ta) + B_I = \varphi(P)f(I)(1-t) - I\left[(1-ta)\Lambda - \frac{tc}{1+r\theta}\right]. \tag{24}$$

This is maximized with respect to I. The solution is only interesting if it yields a positive π_{BD}. The first-order condition yields

$$\varphi(P)f'(I) = \frac{1}{1-t}\left[(1-ta)\Lambda - \frac{tc}{1+r\theta}\right]. \tag{25}$$

The equilibrium after-tax value of the fee is M determined by

$$\pi_{BD}(I) = M\Lambda, \tag{26}$$

which is M plus the advantage (if any, or minus the disadvantage) from the M-related borrowing. It is shown in the appendix that this equilibrium equation together with the definitions given above, but without invoking the first-order condition for optimal I, gives

$$\varphi(X_{BD}) = \left[\varphi(P)f(I)(1-t) + \frac{tcI}{1+r\theta}\right]\frac{\eta}{\Lambda}. \tag{27}$$

This gives the denominator in the formula for the average beta in case B. The beta is a value-weighted average of the betas of the elements of the cash flow X_{BD},

$$\beta_{BD} = \frac{\varphi(P)f(I)(1-t)}{\varphi(P)f(I)(1-t) + \frac{tcI}{1+r\theta}} \cdot \frac{\Lambda}{\eta}\beta_P$$

$$= \left[1 + \frac{\frac{tc}{1+r\theta}}{\varphi(P)\frac{f(I)}{f'(I)I} \cdot f'(I)(1-t)}\right]^{-1} \frac{\Lambda}{\eta}\beta_P. \tag{28}$$

At this point, invoking the first-order condition (25) and introducing the constant-elasticity production function (5) gives the expression

$$\beta_{BD} = \frac{(1-ta)\Lambda - \frac{tc}{1+r\theta}}{(1-ta)\Lambda - \frac{tc}{1+r\theta}(1-\alpha)} \cdot \frac{\Lambda}{\eta}\beta_P. \tag{29}$$

The structure of the expression (16) for case F is easily recognized. So is also the difference between the DRS case and the CRS case: The difference between (29) and (17) resembles the difference between (16) and (10). Again it is clear that β_{BD} is decreasing in the tax rate as long as $c > 0$, and furthermore that $\beta_{BC} < \beta_{BD}$ as long as $tc(1-\alpha)/(1+r\theta) > 0$.

3 Extending the model: Uncertain tax position

The results for case F above are based on the assumption that the firm is certain to be in tax position in period 1. While the tax element tPQ is perfectly correlated with the operating revenue, the depreciation deduction and interest deduction were assumed to be risk free, relying on the firm being in a certain tax position.

Most corporate income taxes have imperfect loss offset. If the tax base is negative one year, there is no immediate refund. The loss may under some systems be carried back and/or forward, but there are usually limitations to this, and the present value is not maintained. In a two-period model a realistic multi-period loss carry-back or carry-forward cannot be represented in detail. An extreme assumption which may be useful as a starting point, and which is meaningful if the two-period model is taken literally and the tax code does not allow carry-backs, is that in these cases, there is no loss offset at all. The cash flow to equity in period 1 is then

$$PQ - B(1+r) - t\chi(PQ - gBr - cI), \tag{30}$$

where χ is an indicator variable, $\chi = 1$ when the firm is in tax position in period 1, $\chi = 0$ if not.[22]

Lund (2002a) arrived at an analytical solution for marginal beta in this case under the assumption that the marginal investment constitutes the whole tax base for the firm.[23] Option valuation techniques were used to find a formula for the value of the uncertain cash flow in period 1, following Ball and Bowers (1983) and Green and Talmor (1985).

At this point it is clear that a marginal beta may now take different meanings. A more realistic marginal beta recognizes that the marginal project is part of a larger activity, and that the probability of being in tax position depends on the outcome of that larger activity. This will be analyzed in line with the model of the previous section: The larger activity consists of a DRS investment project, the output of which is being sold at a single stochastic price in the single future period. An even more realistic model would include more stochastic variables (not perfectly correlated) and/or more periods.

Within this model even the marginal beta will depend on the elasticity α. A lower elasticity means that the probability of being out of tax position is lower. But there is still a difference between the marginal and average beta.

Let case R (for Risky deductions) denote the case with an uncertain tax position, using the simplifications $\eta = 1$, $\theta = 1$ as in case F.[24] The following assumption replaces Assumption 5 above:

Assumption 6: *The tax base in period 1 is operating revenue less cI. When this is positive, there is a tax paid at a rate t. When it is negative, the tax system gives no loss offset at all. There is also a tax relief of taI in period 0. We have $c \in (0, 1]$, $a \in [0, 1]$, and $t[a + c/(1 + r)] < 1$.*

To have c strictly positive is necessary to obtain true uncertainty about the tax position (as long as $\Pr(PQ > 0) = 1$), and for some of the formulae below to hold. The valuation of the non-linear cash flow is specified as follows:

Assumption 7: *A claim to a period-1 cash flow $\max(0, P - K)$, where K is any positive constant, has a period-0 market value according to the model in McDonald and Siegel (1984).* The value can be written as

$$\varphi(P)N(z_1) - \frac{K}{1+r}N(z_2), \tag{31}$$

where

$$z_1 = \frac{\ln(\varphi(P)) - \ln(K/(1+r))}{\sigma} + \sigma/2, \quad z_2 = z_1 - \sigma, \tag{32}$$

N is the standard normal distribution function, and σ is the instantaneous standard deviation of the price.[25]

In what follows it is assumed that the exogenous variables β_P and σ can be seen as unrelated as long as $\sigma > 0$. A change in σ could be interpreted as, e.g., additive or multiplicative noise, stochastically independent of (P, r_m).[26]

It is shown in the appendix that the marginal beta is

$$\beta_{RM} = \frac{1 - ta - \frac{tcN(z_{2D})}{1+r}}{1 - ta}\beta_P, \tag{33}$$

where z_{2D} is given by

$$z_{2D} = \frac{1}{\sigma}\left[\ln\left(1 - ta - \frac{tN(z_{2D})c}{1+r}\right) - \ln(1 - tN(z_{2D} + \sigma)) - \ln\left(\frac{c}{1+r}\right) - \ln(\alpha)\right] - \frac{\sigma}{2}. \tag{34}$$

Although this equation cannot be solved explicitly, it determines (one or more values for) z_{2D} implicitly[27] as function(s) of $t, a, c/(1+r), \sigma$, and α.

Furthermore it is shown that the average beta is

$$\beta_{RA} = \frac{1 - ta - \frac{tcN(z_{2D})}{1+r}}{1 - ta - \frac{tcN(z_{2D})}{1+r}(1-\alpha)}\beta_P. \tag{35}$$

This means that the relationship between marginal and average beta is just as in the previous two cases, which had full certainty about the tax position. There is an extra term containing $tc(1-\alpha)$ subtracted in the denominator.

The two equations (33) and (35) should be compared with (10) and (16). Clearly the effect of the uncertainty in the tax position is similar to a reduced tax rate in period 1, reflecting that the probability of receiving the tax deductions is less than one hundred percent.

For comparison, the marginal beta in the stand-alone CRS case can be found by solving for z_{2C} from the following equation, also shown in the appendix,

$$z_{2C} = \frac{1}{\sigma}\left[\ln\left(1 - ta - \frac{tN(z_{2C})c}{1+r}\right) - \ln(1 - tN(z_{2C} + \sigma)) - \ln\left(\frac{c}{1+r}\right)\right] - \frac{\sigma}{2}, \tag{36}$$

which is the limit of (34) as $\alpha \to 1$. This subcase yields,

$$\beta_{RC} = \left(1 - tN(z_{2C})\frac{c}{(1-ta)(1+r)}\right)\beta_P. \tag{37}$$

This is the case considered in Lund (2002a), except that equation (36) was not given there. Table 1 summarizes the seven subcases considered. The rightmost column gives the ratio of β_i (the beta of the cash flow to equity) to β_P in each subcase i.

To find the derivatives of the betas with respect to $t, a, c/(1+r), \sigma, \alpha$, one can use implicit differentiation of z_{2D} as given in (34) . This is done in the appendix for the case $a = 0$. In order to determine the signs of the derivatives, some restrictions on the parameters are assumed. One basic restriction, cf. footnote 17, is

Assumption 8: *The deductions following a unit investment cost are less than unity in present value terms:* $a + c/(1+r) < 1$.

It is then found that β_{RA} is increasing in σ, decreasing in t, while it may be increasing or decreasing in α, depending on parameters.

A further investigation has been done through numerical solutions of non-linear equations. The purpose of the investigation has been to trace out how the marginal and average betas depend on t, σ, and α.

The numerical investigation has only considered cases with a set to zero, and the ratio $c/(1+r)$ fixed at 1/1.05. The central parameter configuration considered is $t = 0.3, \sigma = 0.3$. These are not unreasonable numbers (when the tax period is one year). The five equity betas for the cases with no borrowing, divided by β_P, are shown in Figure 1 as functions of the scale elasticity α.[28] A sixth relevant curve for comparison would be β_P itself, horizontal at 1.0 in the diagram. This would be the beta of equity without taxation or with true cash flow taxation.

Figure 1 shows that the betas have the expected properties. For simplicity the verbal discussion below will assume $\beta_P = 1$. The two dotted curves show the marginal β when the value of tax deductions is risk free, β_{FC}, and also when a marginal project stands alone with risky deductions (no loss offset), β_{RC}. These do not depend upon the scale elasticity, α. They are both substantially lower than β_P, but the uncertainty of the tax position increases marginal β from 72 percent to 83 percent. This reflects that the depreciation schedule

	a	η	θ	$N(z_2)$	α	marginal vs. average	β_i	β_i/β_P
Case F	$\in [0,1]$	1	1	1	1	marginal	β_{FC}	$\frac{1-ta-\frac{tc}{1+r}}{1-ta}$
					$\in (0,1)$	average	β_{FD}	$\frac{1-ta-\frac{tc}{1+r}}{1-ta-\frac{tc}{1+r}(1-\alpha)}$
Case B	$\in [0,1]$	$\in (0,1]$	$\in (0,1]$	1	1	marginal	β_{BC}	$\frac{(1+r\theta)(1-ta)\Lambda-tc}{(1+r\theta)(1-ta)\Lambda}\cdot\frac{\Lambda}{\eta}$
					$\in (0,1)$	average	β_{BD}	$\frac{(1+r\theta)(1-ta)\Lambda-tc}{(1+r\theta)(1-ta)\Lambda-tc(1-\alpha)}\cdot\frac{\Lambda}{\eta}$
Case R	$\in [0,1]$	1	1	$\in (0,1)$	1	marginal	β_{RC}	$1-tN(z_{2C})\frac{c}{(1-ta)(1+r)}$
					$\in (0,1)$	marginal	β_{RM}	$\frac{1-ta-\frac{tcN(z_{2D})}{1+r}}{1-ta}$
					$\in (0,1)$	average	β_{RA}	$\frac{1-ta-\frac{tcN(z_{2D})}{1+r}}{1-ta-\frac{tcN(z_{2D})}{1+r}(1-\alpha)}$

Equations implicitly defining z_{2C} and z_{2D}:

$$z_{2C} = \frac{1}{\sigma}\left[\ln\left(1-ta-tN(z_{2C})\frac{c}{1+r}\right)-\ln(1-tN(z_{2C}+\sigma))-\ln\left(\frac{c}{1+r}\right)\right]-\frac{\sigma}{2}$$

$$z_{2D} = \frac{1}{\sigma}\left[\ln\left(1-ta-tN(z_{2D})\frac{c}{1+r}\right)-\ln(1-tN(z_{2D}+\sigma))-\ln\left(\frac{c}{1+r}\right)-\ln(\alpha)\right]-\frac{\sigma}{2}$$

Definition of Λ:

$\Lambda \equiv \eta+(1-\eta)\frac{1+r(1-tg)}{1+r\theta} > 0$

Table 1: Beta of equity for the seven subcases, divided by β_P

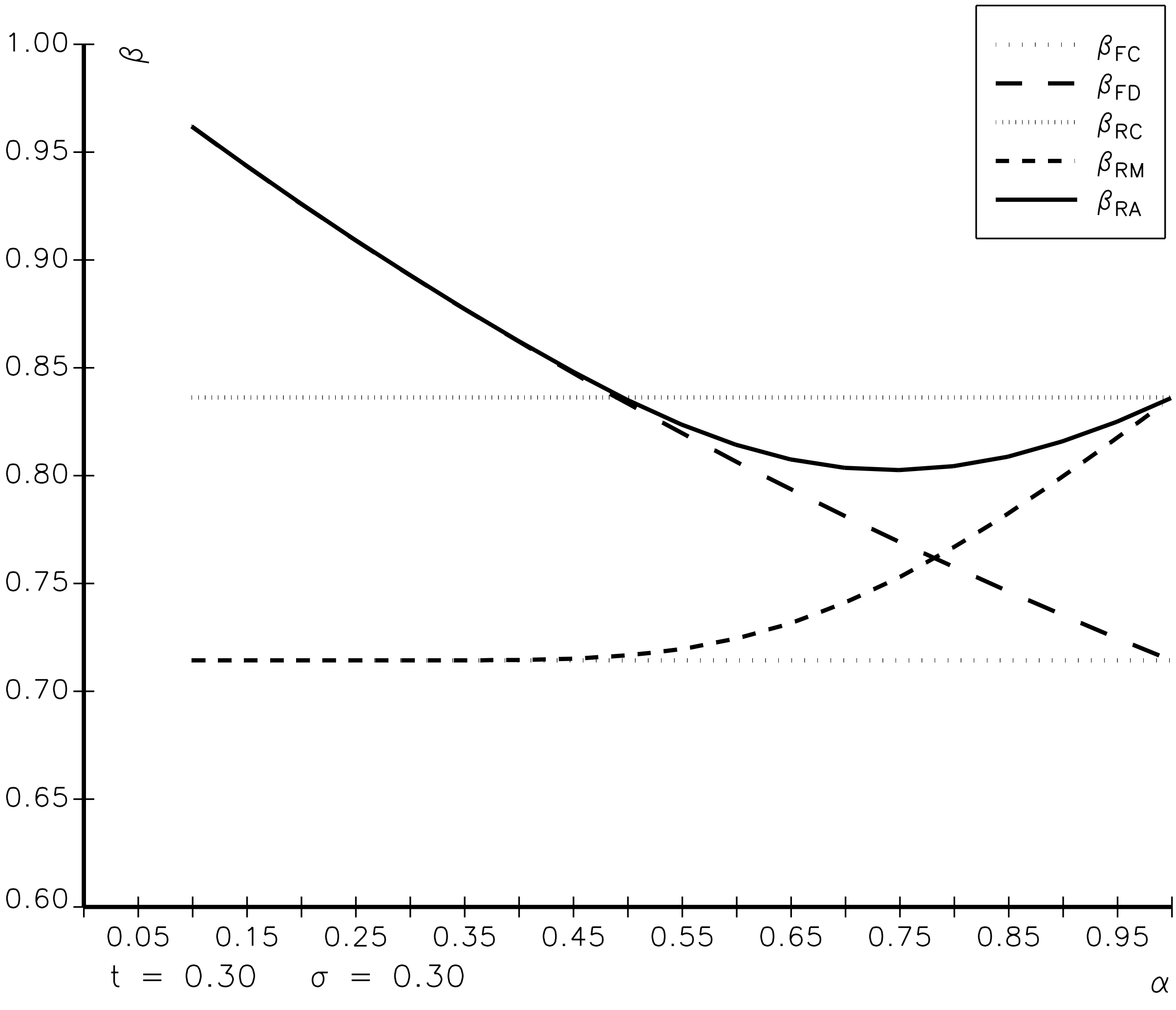

Figure 1: β_i/β_P as functions of scale elasticity, α; $t = \sigma = 0.3, c/(1+r) = 1/1.05$

reduces the systematic risk of equity, and that the uncertain tax position counteracts this to some extent, but not completely.

The two dashed curves show two DRS cases, the average β when deductions are risk free, β_{FD}, and the marginal when they are risky, β_{RM}. The former falls from unity to β_{FC} as α is increased. This simply reflects the difference in tax treatment of M_0 and I, and the fact that M_0 becomes relatively smaller as $\alpha \to 1^-$. β_{RM}, however, rises from β_{FC} to β_{RC} as α is increased. This follows from the increased probability of being out of tax position. When α is low enough, the tax position is virtually certain, and the marginal β under DRS is not different from that under CRS and a certain tax position. But as α increases, so does the uncertainty about the tax position, and as $\alpha \to 1^-$, there is no gain anymore for the marginal project of being taxed together with a DRS project. It approaches the case where the marginal project is taxed alone.

The solid curve shows the average β_{RA} in the DRS case with uncertain tax position. For low α values, the tax position is virtually certain, so there is no discernible difference from β_{FD} of the case of risk free deductions. Then as α exceeds (about) 0.5, the effect of the uncertain tax position is that β_{RA} takes on higher values than β_{FD}, while still being decreasing in α. For even higher α values, however, the curve becomes increasing, as it approaches β_{RM}, which is increasing. This possibility is also seen from the discussion of equation (A29) in the appendix.

Clearly, even the DRS case with risky deductions can have betas substantially lower than β_P. In this case the marginal beta curve, β_{RM}, satisfies the intuition that it has less risk than the stand-alone marginal beta, β_{RC}, as an effect of being taxed together with an infra-marginal cash flow. But the average beta, β_{RA}, does not exhibit this property uniformly, and in fact, the difference between marginal and average beta is just as large in this case as in the case with risk free deductions if only α is low enough.

Figures 2–5 show some sensitivities to changes in the tax rate, t, and the volatility, σ. The three non-constant curves from Figure 1 are reproduced as (similarly) dotted curves, and the corresponding three curves for the new value of t or σ are drawn as dashed or solid. The values of the constant β_{FC} and β_{RC} are now only shown implicitly, as the endpoint values for some of the curves.[29]

Figures 2 and 3 show that all betas are increased if the tax rate is lowered, and vice versa, which was also the main point in Lund (2002a) for the cases considered there. The effect on the lowest values (β_{FC}, which is the limit of β_{RM} for low α, and of β_{FD} for high α) seems to be proportional to $(1 - t)$, which is almost correct when $c/(1 + r)$ is close to unity, cf. equation (10), see also Corollary 2.2 in Lund (2002a). But the higher beta values do not change as much in absolute terms.[30]

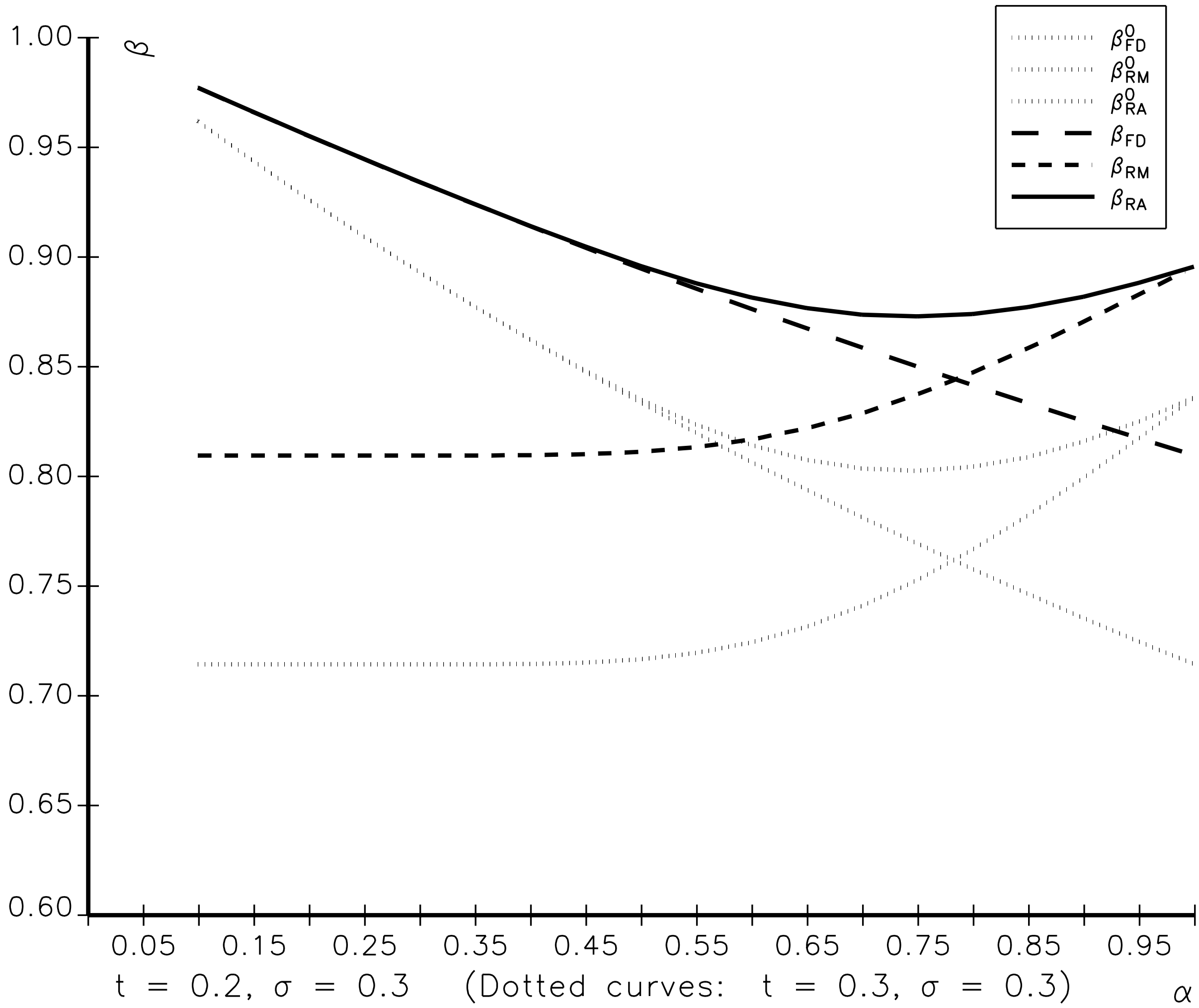

Figure 2: β_i/β_P as functions of scale elasticity, α; varying the tax rate

Figures 4 and 5 show only one β_{FD} curve, as this is unaffected by a change in volatility. The figures show that a lower σ works in the same direction as a higher t, except that β_{FD} is unaffected. But the effects of changes in σ are only discernible for higher values of α, and the magnitudes of the effects are not very large.

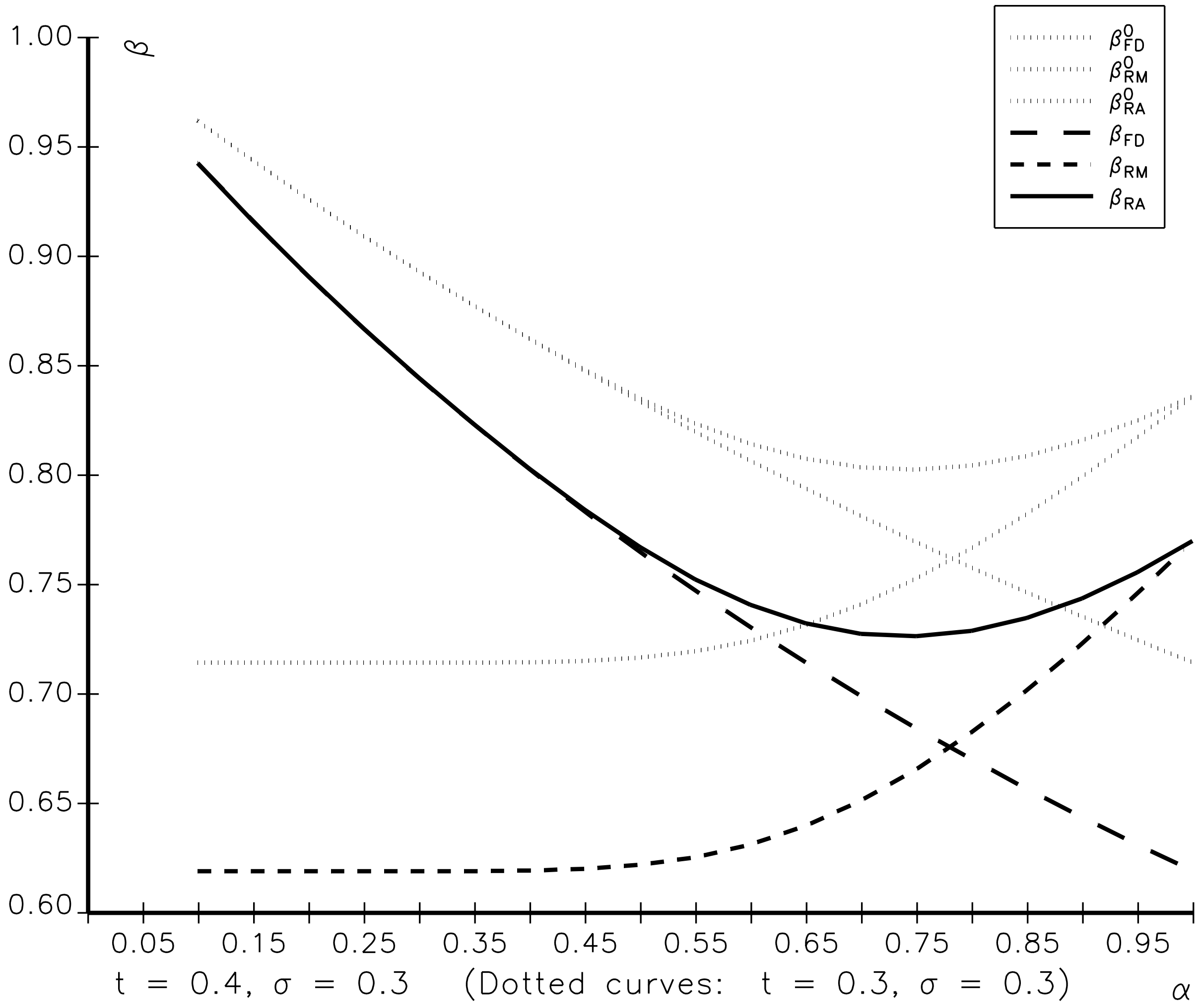

Figure 3: β_i/β_P as functions of scale elasticity, α; varying the tax rate

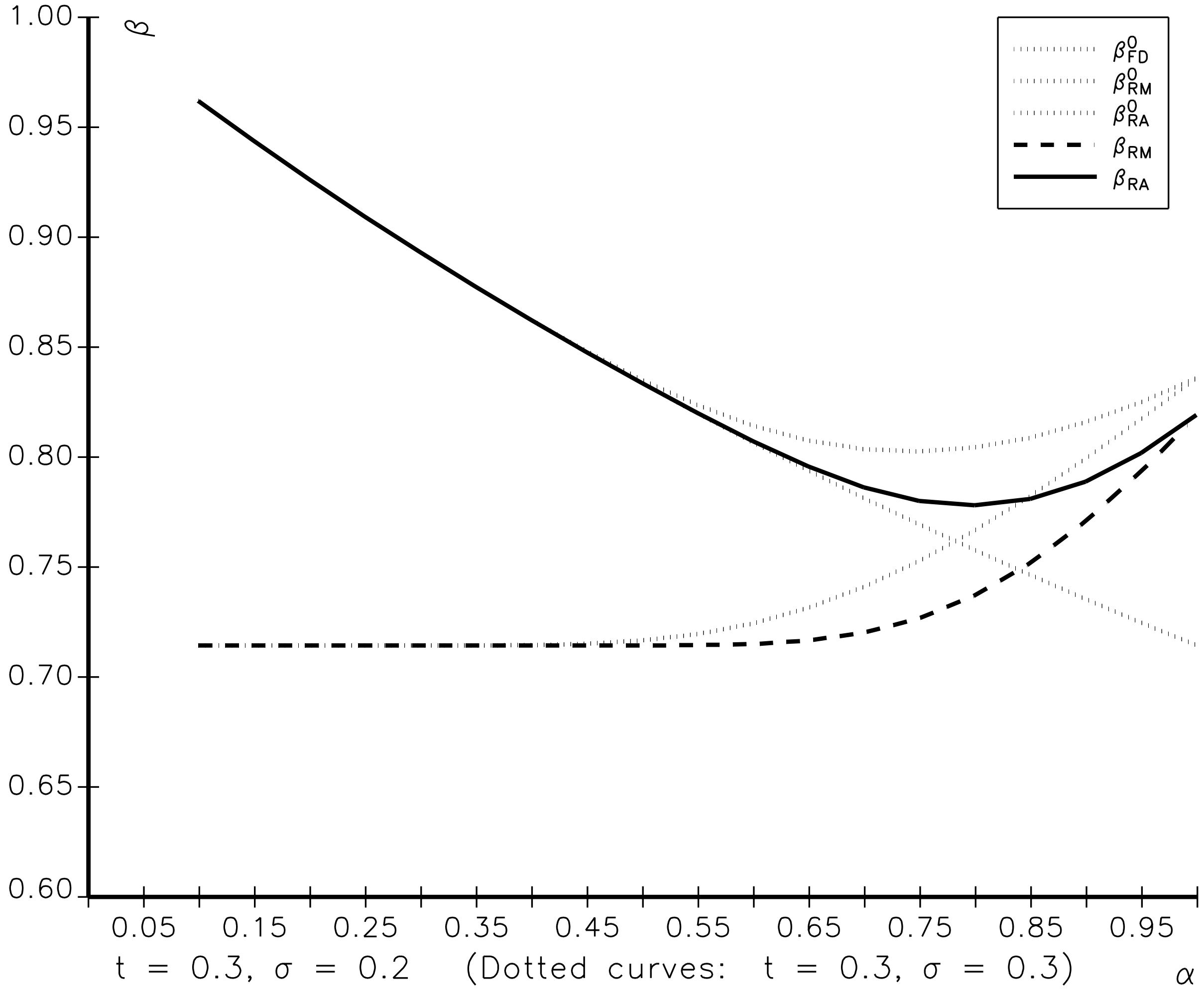

Figure 4: β_i/β_P as functions of scale elasticity, α; varying the volatility

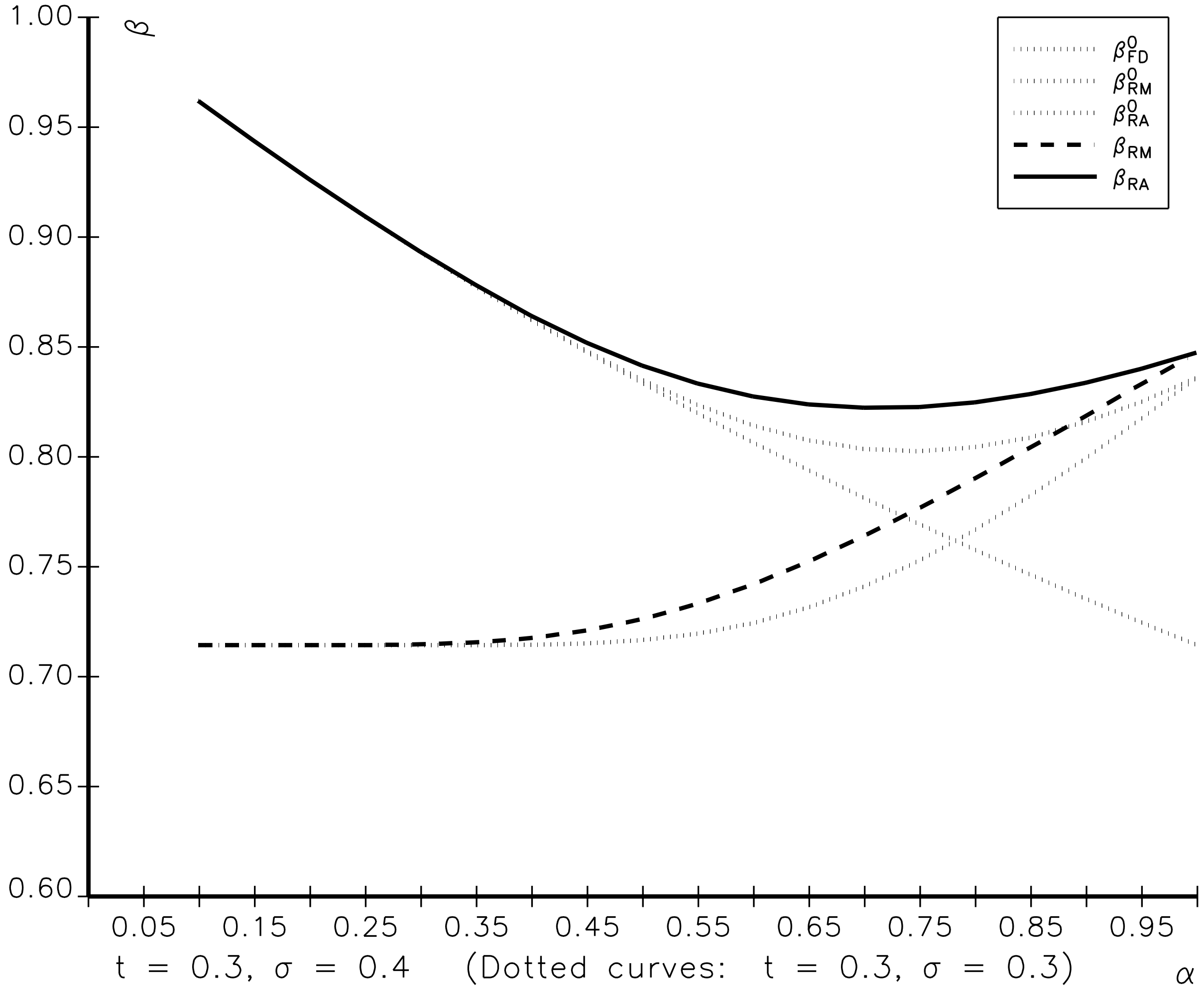

Figure 5: β_i/β_P as functions of scale elasticity, α; varying the volatility

A final question to be analyzed is what happens if the tax treatment of the fee M_0 is changed. The fee is a necessary part of the model in order for the firm's stock to be on the security market line, i.e., for the firm to have a net value of zero. This allows the discussion to consider equilibrium phenomena. At this point different assumptions could be made, but would be ad hoc. However, the tax treatment of M_0 could take different forms, in part because tax systems differ, but in part because M_0 can represent different forms of expenses, such as research and development, purchase of a patent, or purchase of a license. A patent would in many cases be subject to a depreciation schedule, while the other types may be immediately deductible.

A more general model of the tax treatment of M_0 is included in the appendix. It turns out that if the fractions deductible in periods zero and one are identical to a and c, respectively, so that M_0 and I are treated equally, then no effect of decreasing returns to scale remains. The whole investment $M_0 + I$ becomes a marginal investment, and the three betas β_{RC}, β_{RM}, and β_{RA} collapse to the same expression, β_{RC}. This raises the question whether the model of decreasing returns to scale is really only an artifact of the tax system. The answer is that in order to represent this as an equilibrium phenomenon, the fee is needed, and without taxation, the project would be indistinguishable from a marginal project. The intended effect of the model is that the marginal investment within the project is taxed together with the infra-marginal investment. For a project with net value equal to zero, this can probably only be achieved (at least in a two-period model) on the basis of tax discrimination between the expenses incurred in period 0.

4 Cost of capital before taxes

In the previous two sections the effects of the tax system on the beta of equity were analyzed. Via the CAPM equation, (1), this also gives the effects on the cost of equity after corporate taxes, which is reflected in the stock market (but observe the distinction between marginal and average beta).

The cost of capital before corporate taxes, on the other hand, is the traditional measure for the effects of the tax system on the acceptance or rejection of real (non-financial) investment projects. This determines the possible distortionary effects of the tax system, although the present paper does not discuss what would be the relevant basis for comparison under various circumstances. Implicitly the comparison is with a situation without corporate taxation. Also, as stated in the introduction, no general equilibrium effects are considered.

The cost of capital for investment decisions relates to marginal profitability, so the return should be seen in relation to the investment cost I, neglecting the fee M_0. The expected rate of return before corporate taxes, plus 1, is $E(P)Q/I$, which can be rewritten as

$$\frac{E(P)Q}{I} = \frac{E(P)}{\varphi(P)} \cdot \frac{\varphi(P)Q}{I}. \tag{38}$$

Of the two fractions on the right hand side, the first is assumed to be exogenous, and is given by (1) and (3). The second is determined by the requirement under CRS that the project should be marginal after tax, or, under DRS, that its scale should be optimal after tax. For case F above, these requirements are given by (9) and (13), which means that one plus the required expected rate of return before corporate taxes is

$$\frac{E(P)}{\varphi(P)} \cdot \frac{1 - ta - \frac{tc}{1+r}}{1-t}, \tag{39}$$

cf. Hall and Jorgenson (1969), p. 395. The distortion in "one plus the expected rate of return" is the second fraction, which is independent of (total and systematic) risk, only a function of tax parameters and the risk free interest rate. This also holds for the distortion in case B, when borrowing is allowed, which is determined by (25). For both cases the distortion is decreasing in a and $c/(1+r)$. Under Assumption 8 it is increasing in the tax rate in case F. When borrowing is allowed in case F, however, the increasing value of the interest deduction may make the distortion decreasing in the tax rate for sufficiently high leverage.

For case R with an uncertain tax position, the relevant first-order condition is given in equation (A9) in the appendix. One plus the required expected rate of return is

$$\frac{E(P)}{\varphi(P)} \cdot \frac{1 - ta - \frac{tc}{1+r}N(z_{2D})}{1 - tN(z_{1D})}. \tag{40}$$

Again the distortion is independent of systematic risk, but now it depends on total risk through the $N(\cdot)$ expressions.

In the appendix it is shown that when $a = 0$, the endogenous part of expression (40), the second fraction, is increasing in α and σ. A higher α reduces the probability of being in tax position, and thus the expected value of depreciation deductions. This increases distortions. A higher σ has a similar effect, and works additionally by increasing the option value of the authorities' tax claim (i.e., it enhances the effect of the asymmetry), increasing distortions. Furthermore, the distortion goes up if t goes up, and likewise if c goes down, as the direct effects known from (39) are dominating. These results on the cost of capital before taxes should be useful. However, they only formalize what is (more or less) known from before, whereas the results on the cost of capital after taxes are more important as corrections of current knowledge and practice.

5 Discussion

The distinction between an average and a marginal beta is one of the novelties of this paper. It has been shown that this distinction should be made even if the firm pays taxes at the margin with full certainty, given that the tax system treats the fixed cost M_0 and the variable cost I differently. Since uncertainty in the model originates from only one project-related stochastic variable, and since the project without tax has no option(-like) characteristics, there is no difference between marginal and average beta if there are no taxes. But with taxes this distinction appears, even in the simplest case with full certainty about the tax position, if there are decreasing returns to scale.

Whether the distinction between marginal and average beta is important in practice, is another question. Most firms may be happy with a rough estimate of the firm's systematic risk, and may not worry too much about the details determining the required expected rate of return. Since different projects have different risk characteristics in practice, it is impossible to come up with an exact number to be used for a new project. Nevertheless the mechanisms described here should be known by the practitioners, who may then evaluate if, when, and how to take them into account. A thirty percent reduction in beta is hardly negligible.

A seemingly critical assumption in the paper is Assumption 7 on option-like valuation of non-linear cash flows. The underlying assumptions were not detailed, since they are well known. It should be observed, however, that the approach is more general than it seems. It is not necessary to rely on the geometric Brownian motion which is the basis for the standard option valuation theory. Any price process which does not allow arbitrage opportunities will do. But the exact solutions will of course be different with different price processes.

Likewise, Assumption 1 on the CAPM can be relaxed. The crucial assumption is the linear risk measure, which could even be related to more than one factor.

There are of course several limitations of the analysis. The uncertainty is multiplicative, which may not be necessary for the model to work (cf. Lund (2003)), but for the simplicity of the results, in particular in the case of risky deductions. The source of uncertainty is a single stochastic variable in a single period, and there is no carry-forward or carry-back of losses, all of which exaggerates the risk of the deductions. As presented, the model does not allow for risky inflation, the effect of which would depend on the systematic risk of nominally risk free claims. In spite of all this, the model should be a step in the direction of more realism, while retaining the possibility of an analytical solution.

6 Conclusion

Lund (2002a) showed that even in a fully equity financed firm, the beta of equity is decreasing in the tax rate under a typical corporate income tax. The main intuition was that a tax depreciation schedule acts risk-wise in the opposite direction of leverage: It is similar to a loan from the firm to the authorities. In light of this it has been important to consider a more realistic model for the uncertainty of the firm's tax position. The effect of a corporate income tax system on the systematic risk of equity after tax depends critically on loss offset provisions and the probability that the firm will be in tax position in future periods. This will depend on the total activities of a firm. This has been modelled as a decreasing-returns-to-scale technology, which has been acquired at an equilibrium cost, so that the total net value of the activity is zero. The model is a stylized quantification of the claim in, e.g., Gordon and Wilson (1986) and Summers (1987) that the tax value of deductions is close to risk free.

When there are decreasing returns to scale after a sunk cost has been paid, and the sunk cost is immediately tax deductible (such as a license or R&D), while the subsequent investment cost is deductible according to a tax depreciation schedule, there will be a difference between the average and marginal beta of a project. The average beta will be reflected (if the model is true) in the stock market data for the firm's stock, while the marginal beta is relevant for each investment decision within the project. If required rates of return are to be derived from market data, this distinction has to be recognized.

When the firm is not certain to be in tax position at the margin in the future period, the valuation is similar to option valuation. Numerical techniques were used to solve for the systematic risk of equity in these cases. Even in this case the systematic risk of equity is less than the underlying systematic risk (relevant for a no-tax situation), it is decreasing in the tax rate, and increasing in the underlying volatility.

The methods and results demonstrated are crucial for discussions on reforms of corporate income taxation. In particular, the results on after-tax required returns are at odds with current practices. Only if the authorities and firms (and other participants) agree on these methods can there be meaningful discussions. In particular, if firms continue to rely on using required expected rates of return after tax which are fixed irrespective of taxes, there may be beneficial reforms which look bad in the eyes of these firms, cf. Lund (2002b).

Appendix

Derivation of equation (27)

From equation (20) and the various definitions we get

$$\varphi(X_{BD}) = \pi_{BD}(I) + I(1-ta)\eta - \frac{B_M(1+r(1-tg))}{1+r\theta}. \tag{A1}$$

Using the equilibrium condition (26) for M gives

$$\varphi(X_{BD}) - I(1-ta)\eta = M - B_M = \eta M = \frac{\eta}{\Lambda}\pi_{BD}(I). \tag{A2}$$

(Thus we have been able to express $B_M(1+r(1-tg))/(1+r\theta)$ as a fraction of $\pi_{BD}(I)$.) Using the definition (24) of $\pi_{BD}(I)$ gives

$$\varphi(X_{BD}) - I(1-ta)\eta = \left[\varphi(P)f(I)(1-t) + I\frac{ct}{1+r\theta}\right]\frac{\eta}{\Lambda} - I(t-ta)\eta. \tag{A3}$$

This simplifies to equation (27).

Derivation of equations (33)–(37)

This derivation starts with the average beta in case R. In case R the cash flow to equity in period 1 is

$$X_{RD} = Pf(I) - t\cdot\max(Pf(I) - cI, 0). \tag{A4}$$

Under Assumption 7 the valuation, as of one period earlier, of a claim to this is

$$\varphi(X_{RD}) = \varphi(P)f(I) - t\left[\varphi(P)f(I)N(z_{1D}) - \frac{cI}{1+r}N(z_{2D})\right], \tag{A5}$$

where

$$z_{1D} = \frac{\ln(\varphi(P)f(I)) - \ln\left(\frac{cI}{1+r}\right)}{\sigma} + \frac{\sigma}{2}, \tag{A6}$$

and

$$z_{2D} = z_{1D} - \sigma. \tag{A7}$$

The expression in square brackets in (A5) can be rewritten in terms of the standard Black and Scholes' formula for option pricing as $C(\varphi(P)f(I), cI, 1, r, \sigma)$, so that

$$\varphi(X_{RD}) = \varphi(P)f(I) - tC(\varphi(P)f(I), cI, 1, r, \sigma). \tag{A8}$$

The C function has the derivatives $\partial C/\partial(\varphi(P)f(I)) = N(z_{1D})$ and $\partial C/\partial(cI) = -N(z_{2D})/(1+r)$, to be used below.[31]

The firm chooses I to maximize $\pi_{RD}(I) \equiv \varphi(X_{RD}) - I(1-ta)$. The first-order condition is

$$\varphi(P)f'(I) = \frac{\left(1 - ta - \frac{tc}{1+r}N(z_{2D})\right)}{(1 - tN(z_{1D}))}. \tag{A9}$$

Introducing the constant-elasticity production function gives

$$\varphi(P)f(I)(1 - tN(z_{1D})) = \frac{I}{\alpha}\left(1 - ta - \frac{tc}{1+r}N(z_{2D})\right). \tag{A10}$$

Again, M has an equilibrium value equal to π_{RD}, so that the total outlay for a firm to obtain the claim to the cash flow X_{RD} is $M + I(1 - ta) = \pi_{RD} + I(1 - ta) = \varphi(X_{RD})$. The claim is equivalent to holding a portfolio with $f(I)(1 - tN(z_{1D}))$ claims on P, and the rest risk free. The beta is a value-weighted average of the betas of these two elements, i.e.,

$$\beta_{RA} = \frac{\varphi(P) f(I)(1 - tN(z_{1D}))}{\varphi(X_{RD})} \beta_P. \tag{A11}$$

Here, the subscript RA is introduced to show that this is the average beta in case R. By introducing the expression for $\varphi(X_{RD})$ from (A5) and the constant-elasticity production function, this can be simplified as

$$\beta_{RA} = \frac{1 - ta - \frac{tcN(z_{2D})}{1+r}}{1 - ta - \frac{tcN(z_{2D})}{1+r}(1 - \alpha)} \beta_P. \tag{A12}$$

It is also possible to express z_{1D} and z_{2D} in terms of exogenous variables, including the elasticity α, avoiding the decision variables of the firm. Plug in from the first-order condition (A10) into (A6)–(A7) to find

$$z_{2D} = \frac{1}{\sigma}\left[\ln\left(1 - ta - \frac{tcN(z_{2D})}{1+r}\right) - \ln(1 - tN(z_{2D} + \sigma)) - \ln\left(\frac{c}{1+r}\right) - \ln(\alpha)\right] - \frac{\sigma}{2}. \tag{A13}$$

Although it is impossible to solve for z_{2D} explicitly, equation (A13) determines (one or more values of) z_{2D} implicitly as function(s) of $t, a, c/(1 + r), \sigma$, and α.

In order to derive the marginal beta for the same case, consider first the marginal beta derived in Lund (2002a) for the case with an uncertain tax position, equation (24) in that paper. Under the simplifying assumptions $\eta = 1, \theta = 1$, that paper's equation (23) becomes

$$\gamma = \frac{1 - ta - tN(z_{2C})\frac{c}{1+r}}{1 - tN(z_{1C})}, \tag{A14}$$

and the marginal beta can be written

$$\beta_{RC} = \left(1 - ta - tN(z_{2C})\frac{c}{1+r}\right) \beta_P. \tag{A15}$$

The subscript RC (C for CRS) is used here since the case considered in Lund (2002a) did not include the marginal project with some other activity, i.e., as if the case had constant returns to scale.

Again it is possible to express z_{2C} in terms of the exogenous parameters. In this case there is no first-order condition for an interior profit maximum, but the definition of a marginal CRS project, which gives

$$\frac{\varphi(P)Q}{I} = \frac{1 - ta - tN(z_{2C})\frac{c}{1+r}}{1 - tN(z_{1C})}, \tag{A16}$$

cf. equations (5) and (23) in Lund (2002a). This lets us write

$$z_{2C} = \frac{1}{\sigma}\left[\ln\left(1 - ta - \frac{tcN(z_{2C})}{1+r}\right) - \ln(1 - tN(z_{2C} + \sigma)) - \ln\left(\frac{c}{1+r}\right)\right] - \frac{\sigma}{2}, \tag{A17}$$

which, not surprisingly, is the limit of (A13) as α tends to unity. Again, z_{2C} is determined implicitly, this time as function(s) of $t, a, c/(1+r)$, and σ.

What then about the marginal beta for the DRS case? This can be seen as a mixture of the two cases just considered. The marginal beta characterizes a small investment which has a net value of zero. Under imperfect loss offset the value will depend upon the probability of being in tax position. In particular this is crucial in case R, for which it is assumed that after period one there are no more periods, so that the loss cannot be carried forward (nor backward). The criterion for the project being marginal looks similar to (A16), but in this case the valuation of the option-like cash flow to the marginal project in period 1 is based on the risk-adjusted probabilities $N(z_{1D})$ and $N(z_{2D})$, not $N(z_{1C})$ and $N(z_{2C})$, since they should now reflect the probabilities that the whole DRS project is in tax position at the margin. The project which invests I to yield Q, and which is taxed together with the optimally scaled DRS project, is marginal when

$$\frac{\varphi(P)Q}{I} = \frac{1 - ta - tN(z_{2D})\frac{c}{1+r}}{1 - tN(z_{1D})}. \tag{A18}$$

The marginal beta in the DRS case becomes

$$\beta_{RM} = \frac{1 - ta - \frac{tcN(z_{2D})}{1+r}}{1 - ta}\beta_P, \tag{A19}$$

with z_{2D} given from (A13) above.

Partial derivatives of β_{RA}

This section considers the partial derivatives of β_{RA} with respect to the parameters $t, c/(1+r), \sigma$, and α, and determines the signs of these for broad ranges of values of the parameters.

However, in order to restrict the discussion, it will be assumed (a bit further below) that $a = 0$. To simplify the notation, define $\hat{c} \equiv c/(1 + r)$, and in this section write z for z_{2D} defined in (A13).

Implicit differentiation of that equation gives

$$\frac{\partial z}{\partial \sigma} = \frac{1}{\sigma}\left[\frac{-t\hat{c}n(z)\frac{\partial z}{\partial \sigma}}{1 - ta - t\hat{c}N(z)} + \frac{tn(z+\sigma)\left(\frac{\partial z}{\partial \sigma}+1\right)}{1 - tN(z+\sigma)}\right] \tag{A20}$$

$$-\frac{1}{\sigma^2}\left[\ln(1 - ta - t\hat{c}N(z)) - \ln(1 - tN(z+\sigma)) - \ln(\hat{c}) - \ln(\alpha)\right] - \frac{1}{2}, \tag{A21}$$

where $n(\cdot)$ denotes the standard normal density function.

This can be solved for

$$\frac{\partial z}{\partial \sigma} = \frac{-(z+\sigma) + \frac{tn(z+\sigma)}{1-tN(z+\sigma)}}{\sigma + t\left[\frac{\hat{c}n(z)}{1-ta-t\hat{c}N(z)} - \frac{n(z+\sigma)}{1-tN(z+\sigma)}\right]}. \tag{A22}$$

Although difficult to prove analytically, it seems that the numerator is negative, while the denominator is positive. This has been verified numerically for $a = 0$ and $\hat{c} = 1/1.05$, considering a grid of 800 (α, σ, t) vectors, covering the reasonable ranges $\alpha \in [0.1, 1], t \in [0, 0.7], \sigma \in [0.05, 0.5]$. On this basis it is concluded that $\partial z/\partial \sigma < 0$ for reasonable parameter values. The sign of the denominator is also needed below for the sign of the remaining partial derivatives. As verified numerically, it is assumed to be positive:

Assumption 9: *There is no immediate tax relief for investment, i.e., $a = 0$. Moreover,*

$$\sigma + t\left[\frac{\hat{c}n(z)}{1 - ta - t\hat{c}N(z)} - \frac{n(z+\sigma)}{1 - tN(z+\sigma)}\right] > 0.$$

Consider now the partial derivative with respect to α. Using the same method as above, we can show that

$$\frac{\partial z}{\partial \alpha} = \frac{-(1/\alpha)}{\sigma + t\left[\frac{\hat{c}n(z)}{1-ta-t\hat{c}N(z)} - \frac{n(z+\sigma)}{1-tN(z+\sigma)}\right]}, \tag{A23}$$

which is negative under Assumption 9.

Furthermore, we find

$$\frac{\partial z}{\partial t} = \frac{-\frac{a+\hat{c}N(z)}{1-t(a+\hat{c}N(z))} + \frac{N(z+\sigma)}{1-tN(z+\sigma)}}{\sigma + t\left[\frac{\hat{c}n(z)}{1-ta-t\hat{c}N(z)} - \frac{n(z+\sigma)}{1-tN(z+\sigma)}\right]}. \tag{A24}$$

It can be shown that the sign of the numerator is positive if and only if

$$N(z+\sigma) - a - \hat{c}N(z) > 0. \tag{A25}$$

This restriction is implied by Assumption 8, but is somewhat weaker, since $N(z+\sigma) > N(z)$. Under Assumption 8 (or the weaker restriction (A25)) and Assumption 9, we find $\partial z/\partial t > 0$.

Next, we have

$$\frac{\partial z}{\partial a} = \frac{-\frac{t}{1-t(a+\hat{c}N(z))}}{\sigma + t\left[\frac{\hat{c}n(z)}{1-ta-t\hat{c}N(z)} - \frac{n(z+\sigma)}{1-tN(z+\sigma)}\right]}, \tag{A26}$$

which has the same sign as $\partial z/\partial \sigma$ and $\partial z/\partial \alpha$, negative under Assumption 9. The same is true for

$$\frac{\partial z}{\partial \hat{c}} = \frac{-\frac{tN(z)}{1-t(a+\hat{c}N(z))} - \frac{1}{\hat{c}}}{\sigma + t\left[\frac{\hat{c}n(z)}{1-ta-t\hat{c}N(z)} - \frac{n(z+\sigma)}{1-tN(z+\sigma)}\right]}, \tag{A27}$$

which has an additional term compared with $\partial z/\partial a$, since a higher c affects the probability of being in tax position directly, not only via the optimal investment behavior.

Consider now the partial derivatives of β_{RA} from (A12). Let the denominator in (A12) be $D \equiv 1 - t\hat{c}N(z)(1-\alpha)$. Then we find

$$\frac{\partial \beta_{RA}}{\partial t} = \frac{\alpha\hat{c}}{D^2}\left[-tn(z)\frac{\partial z}{\partial t} - N(z)\right]\beta_P, \tag{A28}$$

which is negative under Assumption 9.

Next, we get

$$\frac{\partial \beta_{RA}}{\partial \alpha} = \frac{-t\hat{c}}{D^2}\left[\alpha n(z)\frac{\partial z}{\partial \alpha} + (1-t\hat{c}N(z))N(z)\right]\beta_P. \tag{A29}$$

The sign of this is indeterminate: The expression in square bracket contains two terms of which the first is negative, while the second is positive. As a rough approximation, $(\partial z/\partial \alpha)\cdot\alpha \approx -(1/\sigma)$, so that (since $n(z) < 0.4$) the positive term may dominate if σ is sufficiently large, making the whole equation negative. The numerical illustration in Figures 1–5 shows that the sign of this derivative changes from negative to positive as α increases. However, it can be shown that this does not have to happen when σ is sufficiently large, in which case β_{RA} becomes everywhere decreasing in α.

Next, we find

$$\frac{\partial \beta_{RA}}{\partial \sigma} = \frac{-\alpha t\hat{c}n(z)}{D^2}\cdot\frac{\partial z}{\partial \sigma}\beta_P, \tag{A30}$$

which is positive under Assumption 9.

Finally, there is

$$\frac{\partial \beta_{RA}}{\partial c} = \frac{-t\alpha}{D^2}\left[\hat{c}n(z)\frac{\partial z}{\partial c} + N(z)\right]\beta_P, \tag{A31}$$

which has an indeterminate sign, even under Assumption 9. If σ is not too small, the term containing $N(z)$ will dominate, making the derivative negative.

Partial derivatives of before-tax cost of capital

The cost of capital before taxes in the no-borrowing cases is the exogenous $E(P)/\varphi(P)$ multiplied by

$$\gamma(t, a, \hat{c}, \sigma, \alpha) \equiv \frac{1 - ta - t\hat{c}N(z)}{1 - tN(z+\sigma)}, \tag{A32}$$

given as expression (40) in the main text. As in the previous section of this appendix, the notation is simplified by writing $\hat{c}$ for $c/(1+r)$ and z for z_{2D} defined in (A13). Although not shown in the above equation, z is itself a function of the same five variables, with partial derivatives given in equations (A22)–(A27). Case F with risk free deductions is obtained by letting both $N(\cdot)$ expressions equal to unity, and the CRS case is obtained when $\alpha \to 1$.

It would be interesting to determine the signs of the partial derivatives of this γ function. The simplest expression is found for $\partial\gamma/\partial\alpha$, since α only has an effect via z. At first glance it may seem clear that a higher α leads to a lower z, thus a lower $N(z)$ and a lower $N(z+\sigma)$, which works just as a lower tax rate, reducing γ. But the facts that the arguments of the two $N(\cdot)$ expressions are different, and that N is concave for positive argument values, imply that the reduction in $N(z)$ may exceed the reduction in $N(z+\sigma)$ sufficiently to lead to the opposite effect: It may happen that the reduction in the expected present tax value of the depreciation deduction has the higher impact, not the reduction in the conditional expected present value of the marginal tax rate on the revenue side.

Analytically: The derivative is

$$\frac{\partial\gamma}{\partial\alpha} = \frac{[1 - tN(z+\sigma)](-t\hat{c}n(z)\frac{\partial z}{\partial\alpha}) + [1 - ta - t\hat{c}N(z)]tn(z+\sigma)\frac{\partial z}{\partial\alpha}}{(1 - tN(z+\sigma))^2}. \tag{A33}$$

The numerator can be written as

$$t\frac{\partial z}{\partial\alpha}\left\{[1 - ta - t\hat{c}N(z)]\,n(z+\sigma) - \hat{c}\,[1 - tN(z+\sigma)]\,n(z)\right\}. \tag{A34}$$

Even when $a = 0$ it seems impossible to determine the sign of this expression analytically. While the first expression in square brackets is greater than $\hat{c}$ multiplied by the second, we will have $n(z+\sigma) < n(z)$ as long as $z > -\sigma/2$.

A numerical investigation for $a = 0, \hat{c} = 1/1.05$, and the reasonable intervals $\alpha \in [0.1, 1], t \in [0, 0.7], \sigma \in [0.05, 0.5]$, shows that $\partial\gamma/\partial\alpha > 0$ everywhere (on a grid of 800 points), while $\partial z/\partial\alpha < 0$ (meaning that Assumption 9 is satisfied). This means that the expression in curly braces is negative, due to $n(z+\sigma) < n(z)$.

The economic interpretation is that even though a higher α lowers the probability of being in tax position, also when we take the optimal adjustment in I into consideration, this does not correspond to a uniformly reduced tax rate. The impact on the marginal tax rate on deductions exceeds that on the marginal tax rate on revenue, which implies that the cost of capital is actually increased. (The reason why $N(z+\sigma)$ appears in the formula beside $N(z)$, is that the latter determines the expected value of being in tax position, while the former determines the expected value of P, conditional on being in tax position.)

The difficulty of determining the sign of $\partial z/\partial\alpha$ analytically carries over to the other four first-order partial derivatives, since the expression in curly braces reappears in all of them. Let Γ be the expression in curly braces in (A34), and remember that a numerical investigation has shown that it is negative for the reasonable parameter values which were applied. The partial derivatives are fractions with a positive denominator, as in (A33), so we concentrate on the numerators. In connection with equations (A22)–(A27) it was concluded under Assumption 9 that $\partial z/\partial\sigma < 0$, $\partial z/\partial t > 0$ (under Assumption 8), $\partial z/\partial a < 0$, and $\partial z/\partial\hat{c} < 0$.

The numerator of $\partial\gamma/\partial\sigma$ is

$$t\frac{\partial z}{\partial\sigma}\Gamma + [1 - ta - t\hat{c}N(z)]\, tn(z+\sigma). \tag{A35}$$

From the discussions above it is reasonable to assume that the first term is positive, and the same clearly holds for the second. The whole expression is thus positive.

The numerator of $\partial\gamma/\partial t$ is

$$t\frac{\partial z}{\partial t}\Gamma + [N(z+\sigma) - a - \hat{c}N(z)]\,. \tag{A36}$$

The first term is negative, cf. above. The second is positive (under Assumption 8), so the sign of the total effect is difficult to determine analytically. The numerical investigation showed that $\partial\gamma/\partial t > 0$ everywhere on the grid.

The numerator of $\partial\gamma/\partial a$ is

$$t\frac{\partial z}{\partial a}\Gamma - t\left[1 - tN(z+\sigma)\right]. \tag{A37}$$

The first term is positive, cf. above. The second term, including its minus sign, is negative, so the sign of the total is indeterminate.

The numerator of $\partial\gamma/\partial\hat{c}$ is

$$t\frac{\partial z}{\partial \hat{c}}\Gamma - tN(z)\left[1 - tN(z+\sigma)\right]. \tag{A38}$$

The first term is positive, cf. above. The second term, including its minus sign, is negative, so the sign of the total cannot be determined analytically. The numerical investigation showed that $\partial\gamma/\partial\hat{c} < 0$ everywhere on the grid.

For the parameter ranges investigated, the indirect effects of t and $\hat{c}$ on γ via changes in z counteract the direct effects found from the case of risk free deductions, cf. (39). But the indirect effects are of second order importance, and cannot overturn the direct effects.

More general tax treatment of the fee, M_0

The deductions are bM_0 in period 0, hM_0 in period 1, where b and h are constants in the interval $[0,1]$. The extension of case R will be developed, while the similar extension of case F can be found by setting the probabilities (the $N(\cdot)$ expressions) equal to unity.

In order to distinguish the expressions from those in the main text, this will be called case G (for Generalized tax system). The cash flow to equity in period 1 is

$$X_{GD} = Pf(I) - t\cdot\max(Pf(I) - cI - hM_0, 0). \tag{A39}$$

The valuation, as of one period earlier, of a claim to this is

$$\varphi(X_{GD}) = \varphi(P)f(I) - t\left[\varphi(P)f(I)N(z_{1GD}) - \frac{cI + hM_0}{1+r}N(z_{2GD})\right], \tag{A40}$$

where

$$z_{1GD} = \frac{\ln(\varphi(P)f(I)) - \ln\left(\frac{cI+hM_0}{1+r}\right)}{\sigma} + \frac{\sigma}{2}, \tag{A41}$$

and

$$z_{2GD} = z_{1GD} - \sigma. \tag{A42}$$

Again, the expression in square brackets in (A40) can be rewritten in terms of the standard Black and Scholes' formula for option pricing as $C(\varphi(P)f(I), cI + hM_0, 1, r, \sigma)$, i.e.,

$$\varphi(X_{GD}) = \varphi(P)f(I) - tC(\varphi(P)f(I), cI + hM_0, 1, r, \sigma). \tag{A43}$$

The firm chooses I to maximize $\pi_{GD}(I) \equiv \varphi(X_{GD}) - I(1 - ta)$. From the first-order condition follows

$$\varphi(P)f(I)(1 - tN(z_{1GD})) = \frac{f(I)\left(1 - ta - \frac{tc}{1+r}N(z_{2GD})\right)}{f'(I)}. \tag{A44}$$

Introducing the constant-elasticity production function gives

$$\varphi(P)f(I)(1 - tN(z_{1GD})) = \frac{I}{\alpha}\left(1 - ta - \frac{tc}{1+r}N(z_{2GD})\right). \tag{A45}$$

Equilibrium M_0 is given by

$$M_0(1 - tb) = \varphi(X_{GD}) - I(1 - ta)$$

$$= \frac{I}{\alpha}\left(1 - ta - \frac{tc}{1+r}N(z_{2GD})\right) + \frac{tcIN(z_{2GD})}{1+r} - I(1 - ta) + \frac{thM_0N(z_{2GD})}{1+r}, \tag{A46}$$

which can be solved for

$$M_0 = I\frac{(1-\alpha)[1 - t(a + \frac{cN(z_{2GD})}{1+r})]}{\alpha[1 - t(b + \frac{hN(z_{2GD})}{1+r})]}. \tag{A47}$$

The ratio of the expressions in square brackets in the numerator and the denominator contains the effect of the different tax treatment (if any) of I and M_0, respectively, in risk-adjusted expected present value terms.

We can now solve for $\varphi(X_{GD}) =$

$$\frac{I}{\alpha}\left[1 - ta - \frac{tcN(z_{2GD})}{1+r}(1-\alpha) + \frac{thN(z_{2GD})}{1+r} \cdot \frac{(1-\alpha)[1 - t(a + \frac{cN(z_{2GD})}{1+r})]}{[1 - t(b + \frac{hN(z_{2GD})}{1+r})]}\right]. \tag{A48}$$

This gives the average beta for this case,

$$\beta_{GD} = \frac{1 - ta - \frac{tcN(z_{2GD})}{1+r}}{1 - ta - \frac{tcN(z_{2GD})}{1+r}(1-\alpha) + \frac{thN(z_{2GD})}{1+r} \cdot \frac{(1-\alpha)[1-t(a+\frac{cN(z_{2GD})}{1+r})]}{[1-t(b+\frac{hN(z_{2GD})}{1+r})]}}\beta_P. \tag{A49}$$

Furthermore, we can express z_{2GD} (and thus also $z_{1GD} = z_{2GD}+\sigma$) in terms of the exogenous parameters implicitly through $z_{2GD} =$

$$\frac{1}{\sigma}\left[\ln\left(\frac{1-ta-\frac{tN(z_{2GD})c}{1+r}}{1-tN(z_{2GD}+\sigma)}(1+r)\right)-\ln\left(c+h\frac{(1-\alpha)[1-t(a+\frac{cN(z_{2GD})}{1+r})]}{\alpha[1-t(b+\frac{hN(z_{2GD})}{1+r})]}\right)-\ln(\alpha)\right]-\frac{\sigma}{2}. \tag{A50}$$

Consider now the case where the two expenses in period 0, M_0 and I, are treated equally by the tax system. This amounts to $a = b$ and $c = h$. It can be shown that this makes α vanish from both (A49) and (A50), so that $z_{2GD} = z_{2C}$, and $\beta_{GD} = \beta_{RC}$, found in the main text.

Exact relationship to Lund (2003)

The model in Lund (2003) has exponential decline in both capital, (economic and tax) depreciation, and reinvestment. However, in order to compare with the present paper, reinvestment is excluded by letting $\mu = 0$ in that model, and the number of production periods is reduced to one by setting the depreciation rate $\xi = 1$. That model does not include the parameter θ, so let this paper's $\theta = 1$. By setting this paper's parameters $a = 0,\ g = 1$, and $c = 1$, one obtains the same tax system as in Lund (2003). All tax deductions in Lund (2003) are risk free, so we consider case B in the main text of the present paper, but with $\alpha \to 1$. The equilibrium expected rate of return on a claim on the product price is called ρ and corresponds to $E(P)/\varphi(P) - 1$ in the present paper.

After these specifications of both models, it can be shown that the required expected rate of return before taxes is the same in the two models, when borrowing a fraction $(1-\eta)$ of the financing need is allowed.

The present paper's required expected rate or return before taxes, plus one, can be written as

$$\frac{E(P)Q}{I} = \frac{E(P)}{\varphi(P)}\cdot\frac{\varphi(P)Q}{I} = \rho\cdot\frac{\varphi(P)Q}{I}. \tag{A51}$$

The last fraction in these equations will depend on the financing and the tax system when the marginal investment is chosen optimally. When this is based on the differentiable production function $f(I)$, the fraction can be rewritten as $\varphi(P)Q/I = \varphi(P)f'(I)$, and for the case to be considered, it is found in equation (25) in the main text. Setting

$a = 0, g = 1, c = 1, \theta = 1$, and introducing the definition (18) of Λ, gives

$$\varphi(P)f'(I) = \frac{1}{1-t}\left[\eta + (1-\eta)\frac{1+r(1-t)}{1+r} - \frac{t}{1+r}\right]. \tag{A52}$$

In the notation of Lund (2003), one plus the required before-tax expected rate of return is written as $1 + \rho_b^*$, and through some manipulation of the above equation this can be rewritten as

$$1 + \rho_b^* = (1+\rho)\frac{\varphi(P)Q}{I} = (1+\rho)\left[1 + \frac{tr\eta}{(1+r)(1-t)}\right]. \tag{A53}$$

This should be compared with the result from Lund (2003). With $\mu = 0, \xi = 1$, the weight w in that paper has the value

$$w = \frac{t\eta}{(1+r)[1-t(1-\eta)]}. \tag{A54}$$

When this is entered into the equation for ρ_b^*, one gets

$$\rho_b^* = \frac{(1+r)[1-t(1-\eta)] - t\eta}{(1+r)[1-t(1-\eta)]} \cdot \rho \cdot \frac{1-t(1-\eta)}{1-t} + \frac{tr\eta}{(1+r)[1-t(1-\eta)]} \cdot \frac{1-t(1-\eta)}{1-t} \tag{A55}$$

If 1 is added to both sides, this simplifies to (A53).

Notes

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[1]See, e.g., Hall and Jorgenson (1967, 1969), King and Fullerton (1984), and Sinn (1987).

[2]Modigliani and Miller (1963), Miller (1977).

[3]Gordon and Wilson (1986) write that "depreciation deductions are normally riskfree in nominal terms" (footnote 10, p. 430). Summers (1987) considers the consequences of firms neglecting this fact.

[4]A marginal investment is an investment project which has zero market value if undertaken by the firm.

[5]The CAPM is presented in Assumption 1 and footnote 11 below.

[6]An example with a more general model is Benninga and Sarig (2003).

[7]"In practical capital budgeting, a single discount rate is usually applied to all future cash flows," Brealey and Myers (2003), p. 239. The survey of Graham and Harvey (2001) confirms this.

[8]See the discussion in Lund (2002a) p. 484 and p. 497.

[9]The present paper extends Lund (2002a). Lund (2003) uses a different set of assumptions, which are not in conflict with those of Lund (2002a) or of the present paper. One main difference is that Lund (2003) does not specify a CAPM relationship, only a more general model with value additivity, like in Modigliani and Miller (1963). It is also more general in that it does not assume multiplicative uncertainty, PQ with Q deterministic, as do Lund (2002a) and the present paper. On the other hand, Lund (2003) relies on

very specific multi-period profiles for production, depreciation and borrowing, and does not allow for uncertainty in tax positions or decreasing returns to scale.

[10]The divergence between the results of Derrig (1994) and of the present paper is spelt out in Lund (2001).

[11]Of course, r_i is the rate of return of shares in firm i, r is the riskless interest rate, r_m is the rate of return on the market portfolio, $\beta_i \equiv \operatorname{cov}(r_i, r_m)/\operatorname{var}(r_m)$, and E is the expectation operator. The original model is derived in Sharpe (1964), Lintner (1965), and Mossin (1966).

All variables are nominal. As long as the tax system is based on nominal values, the model is only consistent with a rate of inflation which is known with certainty, and fixed exchange rates. The underlying real CAPM would then be

$$\frac{1+E(r_i)}{1+\dot{p}} = \frac{1+r\theta}{1+\dot{p}} + \beta_i \left[\frac{1+E(r_m)}{1+\dot{p}} - \frac{1+r\theta}{1+\dot{p}}\right],$$

where $\dot{p}$ is the rate of inflation.

[12] A tax-adjusted CAPM appears, e.g., in Sick (1990) or Benninga and Sarig (1997, 2003).

[13]In Bulow and Summers (1984) it is assumed that even a change in the U.S. corporate tax system can be meaningfully analyzed with partial equilibrium methods, since the firms traded in the U.S. capital market have their activities world wide, and the "U.S. corporate sector represents less than one-tenth of the world free market wealth" (their footnote 3).

[14]The product price has what McDonald and Siegel (1984) call an (expected-)rate-of-return shortfall.

[15]η is in this sense the ratio of equity to assets after tax, as will become clear below.

[16]Another defence for not introducing a more sophisticated theory of debt financing is that the survey by Graham and Harvey (2001) shows that most firms have a fixed target debt ratio.

[17]In parts of the literature, such as King (1977), p. 232, the nominal sum of deductions, here $a + c$, is set to unity. But there "is not need to restrict the sum" of deductions "to unity," according to King and Fullerton (1984), p. 19, who observe that at "certain times it exceeds unity (for example, when accelerated depreciation does not reduce the base for standard depreciation allowances)." In the present paper, a and c are considered as separate, exogenous variables, so that an increase in a is analyzed as if c is kept constant, and vice versa.

[18]The alternative valuation by elements is known as adjusted present value, APV, since Myers (1974).

[19]To "unlever" betas is known from standard textbooks. See Lund (2002a, pp. 484, 497) on "untaxing" betas from market data when the firm operates under different tax systems, and on the mistakes which can be made if firms or authorities evaluate different tax systems using a single required expected rate of return.

[20]If $g = 1$ (interest payments are fully tax deductible) debt may be attractive for tax reasons, but if at the same time $\theta = 1 - tg$, this exact advantage is captured in the capital market where the firm's stock is traded, so that the firm is indifferent towards borrowing.

[21]For simplicity, the model does not allow for different leverage of the two costs M and I, although that might have been more realistic.

[22]Partial loss offset may have to do with loss of (expected) present value due to (the possibility of) postponed deductions. If a loss can only be deducted within the same period, however, a concept of partial loss offset would only be interesting if the project is of non-negligible size and the firm has other activity, since without other activity, the tax base from the project is either positive or negative — no third alternative exists. For analyzing the case with other activity, simplify here by setting $B = 0$. Partial loss offset occurs in two cases: If $PQ - cI < 0$ and the net taxable income from the other activity is in the interval $(0, |PQ - cI|)$, then the project loss reduces the taxes due on the other activity. If $PQ - cI > 0$ and the net taxable income from the other activity is in the

interval $(-(PQ - cI), 0)$, then deduction of the loss from the other activity reduces taxes due on the project.

[23]The current paper improves upon the solution for the case considered in Lund (2002a), by pointing out that the variables z_1 and z_2 used in equation (19) in that paper can be rewritten in terms of the exogenous parameters, given that the production function has a constant elasticity. Observe in particular that whereas the option value in general depends on a rate-of-return shortfall (in an unconstrained equilibrium often identified as a convenience yield), this dependency is not reflected in the formulae here, given that the first-order condition of the firm is satisfied.

[24]It would certainly be interesting to consider leverage and uncertain tax positions in the same model. However, the analysis which follows, and in particular the extension to more general tax deductions in the appendix, is sufficiently complicated as it is. After all, it is well known that the beta of equity depends on the tax rate in the presence of leverage, and one main point of Lund (2002a) and the present paper is to establish that this also holds in the absence of leverage. Two starting points for extending this research to combine leverage and an uncertain tax position are Galai (1988) with risky debt and Lund (2002a) with default-free debt. Assumptions 3 and 5 in Lund (2002a) may be somewhat unrealistic when taken together. They imply that the debt is repaid with certainty, for instance by a parent company if the firm analyzed has insufficient cash flow. At the same time the tax value of the interest deduction is only obtained to the extent that the firm is in tax position. One could imagine instead that the parent company also obtains an interest deduction, but perhaps at a different tax rate.

[25]Lund (2002a) discusses the conditions under which the formula can be modified with $r\theta$ replacing r. If $\theta < 1$, this must rely on an assumption that anyone who trades in securities is more heavily taxed on their interest income than on their equity income. However, in order to simplify the presentation of the additional complications of this section, θ is set to unity in what follows.

[26]Specifically, additive noise could be ε, stochastically independent of (P, r_m), with $E(\varepsilon) = 0$. Then $\text{cov}(P + \varepsilon, r_m) = \text{cov}(P, r_m)$. Multiplicative noise could be ψ, stochastically independent of (P, r_m), with $E(\psi) = 1$. Then $\text{cov}(P\psi, r_m) = \text{cov}(P, r_m)$. These reasonable cases show that the correlation coefficient of (P, r_m) should not (always) be assumed unaffected if P becomes more risky. This points out an important reservation to the discussion in Dixit and Pindyck (1994), in particular the claim on p. 179 that "when the σ of the P asset increases, μ must increase" (where μ is the required expected rate of return according to the CAPM). This is not true in general (but it can be true due to non-linear taxation). Of course, the argument made here does not mean that σ could be zero while β is different from zero.

[27]The solutions (when $a = 0$ and $c/(1 + r) = 1/1.05$) show that z_{2D} is negative (or in one case equal to 0.001) approximately when $\sigma - t - 2(1 - \alpha) \geq 0.3$, an (approximate) linear relationship. To interpret this, recall that $N(z_{2D})$ is the risk-adjusted probability of being in tax position. This will be less than 0.5 if σ and α are large, while t is small. Since the formula relies on the first order condition, the effect via optimal investment is included here. Optimal investment increases when t is small, even though this increases the probability of being out of tax position at the margin. A high α reduces the infra-marginal profit. The effect of an increased σ may not be so obvious, since the probability of P being less than its median is unaffected (and equal to 0.5). But the tax base also includes a negative constant term, which explains the effect.

[28]For each numerical version of each of the nonlinear equation systems (34) and (36) the solution method identified one solution which might not be a unique solution. The program then did a grid search through 400 z_2 values for other solutions, but these were never found. It seems reasonable to conclude that the solutions found are likely to be unique. For equation (36) the uniqueness may depend on α being a constant. A more general version with a non-constant $\alpha(I)$ might lead to several solutions.

[29]So far no indications have been found that the dependency on t or σ should be non-monotonous. But these are solutions to non-linear equations, and the possibility has not

been ruled out. For the parameters shown, however, there is every reason to believe that the solutions are unique.

[30] This effect of a different tax rate becomes particularly pronounced under some systems of taxation of natural resource rents, in which marginal tax rates on firms have been between 50 and 85 percent. When these systems have investment based deductions spread over several years, such as depreciation deductions, the beta of equity becomes quite low. Serious mistakes could be made if firms apply the same cost of equity under such tax systems as under others, cf. Lund (2002b).

[31] The partial derivatives of Black and Scholes' formula can be found, e.g., in Haug (1998), or in most textbooks on option theory. They look as if they neglect the dependence of z_1 and z_2 on the arguments, but they do not.

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