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A NOTE ON THE (IN)STABILITY OF DIAMOND´S GROWTH MODEL

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Abstract

Diamond's two-period OLG growth model is based on the assumption that the stock of capital in any period is equal to the wealth accumulated in the previous period by the generation of pensioners. This stock equlibrium condition may appear an innocuous paraphrase of the ordinary macro-economic flow equilibrium condition, S = I.

This is not the case. In this note I demonstrate that Diamond's solution is unstable in a monetary market economy where households and firms make independent decisions as to how much to save and how much to invest. An increase in the rate of interest above the Diamond long-run equilibrium level will cause saving to fall by more than investment and, hence, result in excess demand for loanable funds and an upward pressure on the rate of interest.

However, substituting the ordinary S = I flow equilibrium condition for Diamonds stock equilibrium condition reveals that the model has another solution - the rate of interest equals the rate of growth - and that this solution is stable in a capital-based economy (contrary to the pure consumption loan model of interest suggested by Samuelson(1958)).

The model has interesting implications. Diamond's model predict that an increase in rate of time preference causing the young generation to save less will reduce the capital stock and raise the rate of interest. However, the S = I based two period OLG model reveals that the old generation's consumption falls by more than the the young generation's consumption increases. Consequently, excess supply of loanable funds will drive down the rate of interest. If the rate of interest is equal to the rate of growth an increase in the time preference has no effect on the supply of loanable funds and, consequently, neither on the rate of interest or the stock of capital. Whether people prefer to consume as young or old should not be a matter of public concern (although the transition from one state to another may be).

1 Introduction

The Diamond (1965) growth model has become a classic. Today, 40 years after the publication of "National debt in a neoclassical growth model" we find the model reproduced in most text-books of growth theory and advanced macroeconomics. Not exactly in the somewhat obscure form in which it was originally presented by Diamond. It has become distilled into a simple easy-to-understand textbook model which leaves aside the questions that triggered Diamond.

Diamond's problem was to square the insights provided by Samuelson's "An exact onsumption-loan model of interest with and without the social contrivance of money" from 1958 - the first OLG-mode (?) - and Phelps' Solow-inspired "Golden Rule of Accumulation: A Fable for Growthmen" from 1961. The problem was that the solution of the Samuelson-model - the rate of interest equals the rate of growth - was found to be unstable (without some kind of social contract), and that Phelps' advice to growthmen who wanted to attain the maximum sustainable per capita consumption - i.e. equate the saving and the capital income ratios - hinged on economic policy to pursue this goal. That triggered three questions: What is the long-run (stable) rate of interest in a capital-based market econommy? Might it be equal to the socially optimal rate, i.e. the rate of growth? What is the impact on the rate of interest (and the stock of capital) of national debt?

In this note I pose the daring assertion that Diamond's analysis is flawed. Diamond's analysis is based on the implicit assumption that the households have access to no other temporary abode for purchasing power but real (productive) capital and, consequently in a two-period OLG setting, that the stock of capital available to the young generation must be equal to the previous generation's saving. Apparently, Diamond assumes that this (stock) equilibrium condition is equivalent to the macroeconomic equilibrium condition that current supply equals current demand or, as the classical economists put it, that the supply of loanable funds equals the demand for loanable funds.

As I demonstrate in this note, substituting the (flow) equilibrium condition for the (stock) equilibrium condition turns his results upside down. The equilibrium rate pof interest derived by Diamond is found to be unstable. However, the model has two roots and the other (an stable) one is the socially optimal one, cf. the rate of interest equals the rate of growth.

2 A simplified version of the Diamond model

As Samuelson, Diamond considers a two-period OLG model. Each generation works in one period and live as pensioners from their saving for yet another period. The problem of the representative household is to determine how much of its first-period (wage)income, w, it should save for consumption in the second period. The problem of the representative firm is to determine the profit maximizing capital stock per employee.

To make things as simple as possible (and true to both the Samuelson model and the original version of the Diamond model) we shall assume that

- Each new generation is born "naked" (i.e. that they have no initial wealth)
- The rate of growth of the population (n) is exogenous
- There is no technological progress
- Households' utility function is log-linear
- Firms face a C-D production function
- No uncertainty

2.1 Household behavior

A household 'born' in period t maximizes its utility function

$$U = \ln(c_{1t}) + \frac{1}{1+\rho} \cdot \ln(c_{2t})$$

subject to its budget constraint

$$c_{1t} + c_{2t} \cdot \frac{1}{1 + r_{t+1}} \leq w_t$$

The notation (and presentation of the solution) follow Romer (2001) rather than the original Diamond (1965) paper:

c_{jt} - consumption i period j, $j = (1, 2)$

r_{t+1} - the rate of interest in period $t+1$, i.e. the return in period $t+1$ from saving in period t

ρ - rate of time preference (constant)

The solution (f.o.c.) is

$$c_{1t} = \frac{1+\rho}{2+\rho} \cdot w_t \tag{1}$$

$$c_{2t} = \frac{1+r_{t+1}}{2+\rho} \cdot w_t \tag{2}$$

$$s_{1t} = w_t - c_{1t} = \frac{1}{2+\rho} \cdot w_t \tag{3}$$

2.2 Firm behavior

A representative firm maximizes profit per worker

$$\pi_t = y_t - k_t \cdot r_t$$

with respect to the stock of capital per worker, k_t, subject to the production function

$$y_t = k_t^{\alpha} \tag{4}$$

The solution (f.o.c) is

$$k_t = \left(\frac{\alpha}{r_t}\right)^{\frac{1}{1-\alpha}} \tag{5}$$

from which

$$y_t = \left(\frac{\alpha}{r_t}\right)^{\frac{\alpha}{1-\alpha}} \tag{6}$$

$$k_t = \alpha \cdot \frac{y_t}{r_t} \tag{7}$$

2.3 Diamond's solution

Diamond solves the model by imposing the equilibrium condition that the stock of capital in period t+1 must be equal to the young generation's saving in the previous period. Normalizing the size of the population in period t to one Diamond's (stock) equilibrium condition reduces to

$$k_{t+1} = \frac{s_{1t}}{1+n}$$

By expressing s_{1t} and k_{t+1}as functions of, respectively, r_t and r_{t+1}, he derives the equation of motion of r

$$r_{t+1} = \left[\frac{\alpha}{1-\alpha} \cdot (2+\rho) \cdot (1+n)\right]^{1-\alpha} \cdot r_t^{\alpha} \tag{8}$$

depicted below

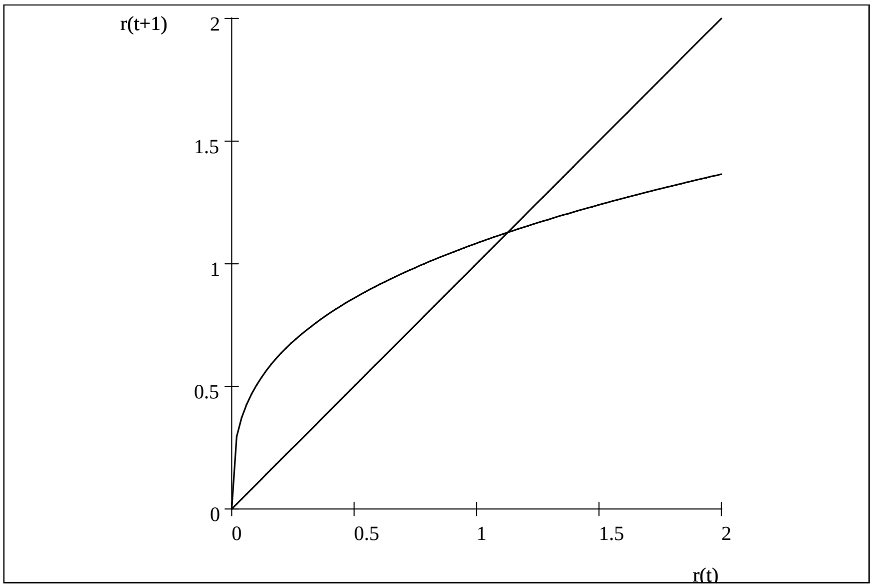


Figure 1: The equation of motion of r (for $\alpha = 0.3333, n = 0.10,\ \rho = 0.05$)

The steady state solution is

$$k^* = \left(\frac{(1-\alpha)}{(2+\rho)\cdot(1+n)} \right)^{\frac{1}{1-\alpha}} \tag{9}$$

$$r^* = y'(k^*) = \frac{\alpha}{1-\alpha} \cdot (2+\rho)\cdot(1+n) \tag{10}$$

Diamond concludes that the equilibrium solution is unique and stable, and that the economy will move towards the golden rule steady state only in the highly unlikely case that

$$n = \frac{\alpha \cdot (2+\rho)}{1-\alpha\cdot(3+\rho)} \tag{11}$$

2.4 Romer's solution

Romer derives the equilibrium solution by expressing s_{1t} as a function of k_t which result in the following equation of motion of k

$$k_{t+1} = \frac{s_{1t}}{1+n} = \frac{(1-\alpha)}{(2+\rho)\cdot(1+n)} \cdot k^{\alpha} \tag{12}$$

depicted in figure 2 below

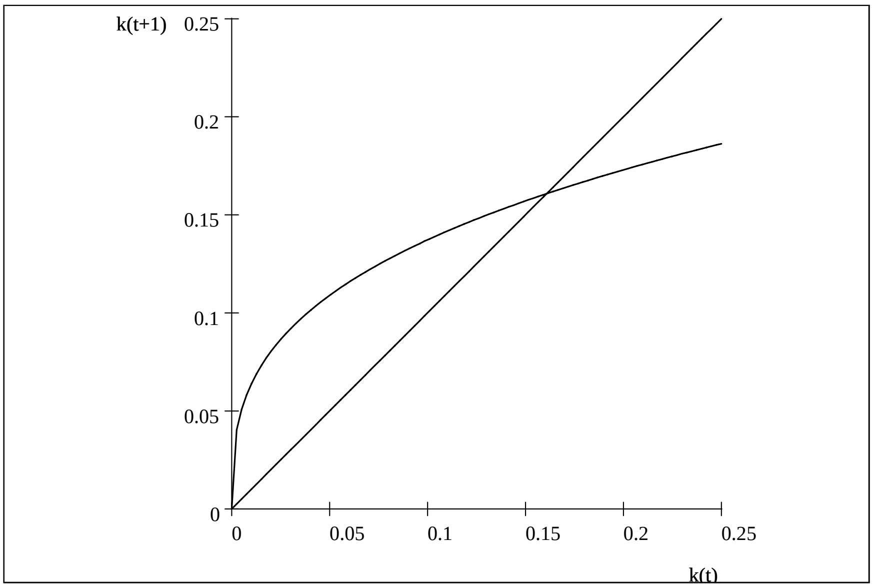


Figure 2: The equation of motion of k (for $\alpha = 0.3333, n = 0.10,\ \rho = 0.05$)

The steady state solution is, of course, the same as derived above.

For the reference parameter values in this paper ($\alpha = 0.3333, n = 0.10$, $\rho = 0.05$) we get $k^* = 0.161, r^* = 1.13$

Surprisingly and unlike Samuelson (1958), neither Diamond, nor Romer comment on the economic sense of the equilibrium solution of the model. For $\alpha = 0.3333, n = 0,\ \rho = 0$, i.e. when no of the three causes of interest suggested by Böhm-Bawerk (Lutz, 1968) apply, the model suggests an equilibrium rate of interest of 1 (=100 percent)! What is the intuition behoind that result?

2.5 An alternative solution

It appears surprising that Diamond (contrary to Samuelson) imposes the constraint that the stock of capital must be equal to the wealth of households rather than the ordinary macroeconomic equilibrium condition

$$S = I$$

Apparently Diamond assumes that the (stock) equilibrium conditionis just an innocuous paraphrase of the common macro-economic (flow) equilibrium condition. However, that is a serious mistake. The two equilibrium conditions are not identical and substituting the flow eaqilibrium conditoin for the (stock) equilibrium condition produces very different results.

By definition national saving,S_t, is equal to income (GDP) minus consumption

$$\begin{aligned} S_t &= y_t - c_{1t} - \frac{c_{2,t-1}}{1+n} \\ &= y_t - \frac{1+\rho}{2+\rho} \cdot (1-\alpha) \cdot y_t - \frac{1+r_t}{(2+\rho)\cdot(1+n)} \cdot (1-\alpha) \cdot y_{t-1} \end{aligned} \tag{13}$$

Investment, $I_{t,}$ is by definition equal the addition to the capital stock in period t, i.e.

$$I_t = \alpha \cdot \frac{y_{t+1}}{r_{t+1}} \cdot (1+n) - \alpha \cdot \frac{y_t}{r_t} \tag{14}$$

By substituting r^* for r_{t+j} and y^* for y_{t+j}, $j = (-1, 0, 1)$ get the following equilibrium condition

$$1 - (1-\alpha) \cdot \left[\frac{1+\rho}{2+\rho} + \frac{1+r^*}{(2+\rho)\cdot(1+n)}\right] = \alpha \cdot \frac{n}{r^*} \tag{15}$$

The equation has two roots

$$r^* = n; \quad r^* = \frac{\alpha}{1-\alpha} \cdot (2+\rho) \cdot (1+n) \tag{16}$$

of which the first one corresponds to the solution of Samuelson's pure consumption loan model of interest rate and the second one to the solution suggested by Diamond and Romer.

2.6 Stability analysis

The fact that we find two solutions if we reject the unrealistic assumption that household wealth must be equal to the value of productive capital poses the question: Are the solutions stable?

The answer to this question might have appeared obvious to Diamond. At the time he wrote his paper it was well-known that the neat $r = n$ solution of Samuelson's pure consumption loan model was unstable (at least without some kind of social contract) and regarded as an intellectual curiosity.

This result does not carry over in (our simplified version of) the Diamond model (as, Diamond apparently assumed).

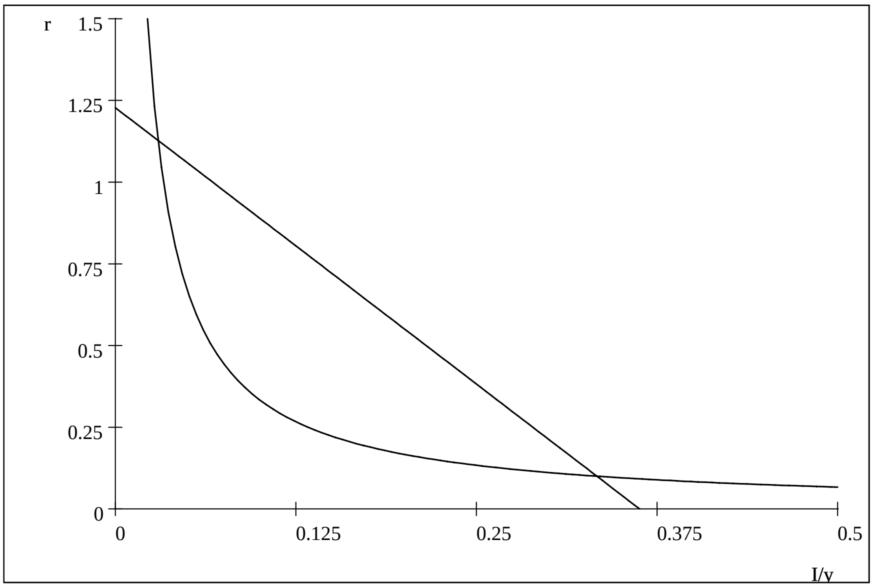


Figure 3: The ratios of investment and saving to income as a functions of the rate of interest (for $\alpha = 0.333$, $n = 0.1$, $\rho = 0.05$, $\gamma = 0$, $\tau = 0$)

Figure 1 illustrates why. The S/y is a linear function in r and $I/y = f(r)$ is a hyperbola. Consequently, (if the model has a solution), the S/y and I/r schedules must intersect twice (if the two solutions don't coincide)[1]. The lower solution is stable (as $\partial(S - I)/\partial r > 0$, indicating an excess supply of loanable funds and a downward pressure on the rate of interest) and the higher solution is unstable (as $\partial(S - I)/\partial r < 0$,indicating an excess demand for loanable funds and an upward pressure on the rate of interest). In the Samuelson model there

[1] Diamond (p. 1132 and 1132) draws graphs indicationg that the saving function is linear and the capital demand curve convex from origo, but curiously he drawings indicate just one solution that might be stable (if the saving function is steeper than the capital demand function) or unstable if the capital demand function is the steeper one.

is no productive capital. The downward sloping saving function implies that $\partial(S - I)/\partial r < 0$ for all values of r and, consequently, that the solution is unstable.

Which of the two roots in our model is the lower one? In the simple Diamond model analyzed here, there can be little doubt, that - contrary to what Diamond assumed - $r^* = n$ is the stable root. For $r^* = \frac{\alpha}{1-\alpha} \cdot (2+\rho) \cdot (1+n)$ to be stable we must have that

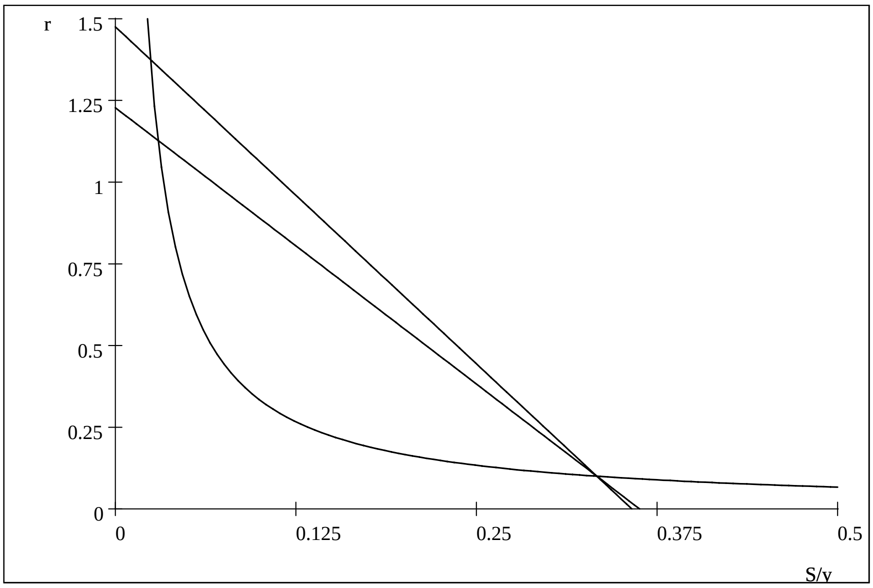
$$n > \frac{\alpha \cdot (2+\rho)}{1 - 3 \cdot \alpha - \rho \cdot \alpha}$$

which appears inconceivable for any realistic value of α[2]. (To see that, assume that the capital income share takes a value in the neighborhood of $\frac{1}{3}$.)

2.7 Implications

2.7.1 The effect of an increase in the time preference (ρ)

In Diamond's model an increase in the time preference reduces the young generation's saving and, consequently, the stock of capiotal for the next generation. The rate of interest increases. However, if $r^* = n$, then the time preference has no effect on the rate of interest, the stock of capital and total income.



[2] Diamond considers only one root, $r^* = \frac{\alpha}{1-\alpha} \cdot (2+\rho) \cdot (1+n)$.As he (contrary to Samuelson) 'assumes' that the economy is stable, that root must be stable! He does not address the problem that the rate of interest should be in the order of magnitude of 100 percent in case of no growth and no time preference (Böhm's first and second cause of interest).

Figure 4: The effect of an increase in the rate of time preference, ρ, from 0.05 to 0.5

As illustrated in figure 2, the only effect of an increase in ρ is to rotate the S/y schedule clockwise around the equilibrium, $r^* = n$, $S/y = \alpha$. The saving ratio falls if $r^* < n$ and increases if $r^* > n$. That may sound counter-intuitive. However, the logic is clear enough: The young generation saves less, but the reduction in the older generation's consumption is even larger, so total saving increases.

2.7.2 The effect of fiscal policy

Suppose that the government increases public consumption by a constant fraction of y, $G = \gamma \cdot y$, and raises taxes by a constant fraction of y, $T = \tau \cdot y$. Public expenditures and taxes have no effect on the I/y schedule, but affect the location and/or slope of the S/y schedule,

$$S/y = \left[1 - (1-\alpha) \cdot \left(\frac{1+\rho}{2+\rho} + \frac{1+r^*}{(2+\rho)\cdot(1+n)}\right)\right] \cdot (1-\tau) + (\tau - \gamma) \qquad (17)$$

An isolated increase in γ causes a parallel shift of the S/y schedule towards the left. The lower (stable) root rises and the higher (unstable) root falls (figure 5a). This is what we would expect. However, Diamond is in troubles, he only gets the same (qualitative) result by assuming that $r^* = \frac{\alpha}{1-\alpha} \cdot (2+\rho) \cdot (1+n)$ is stable.

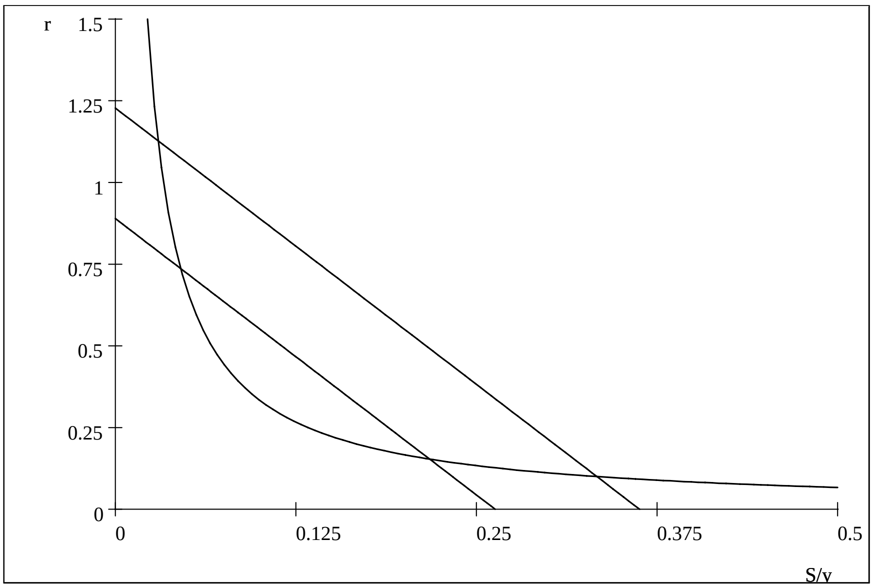


Figure 5a: The effect of an increase in public expenditures ($\gamma = 0.1$)

A balanced budget expansion, $\gamma = \tau > 0$ causes a clockwise rotation of the S/y schedule (the schedule gets steeper). The lower (stable) root rises and the higher (unstable) root falls, but by less than in the former case (figure 5b).

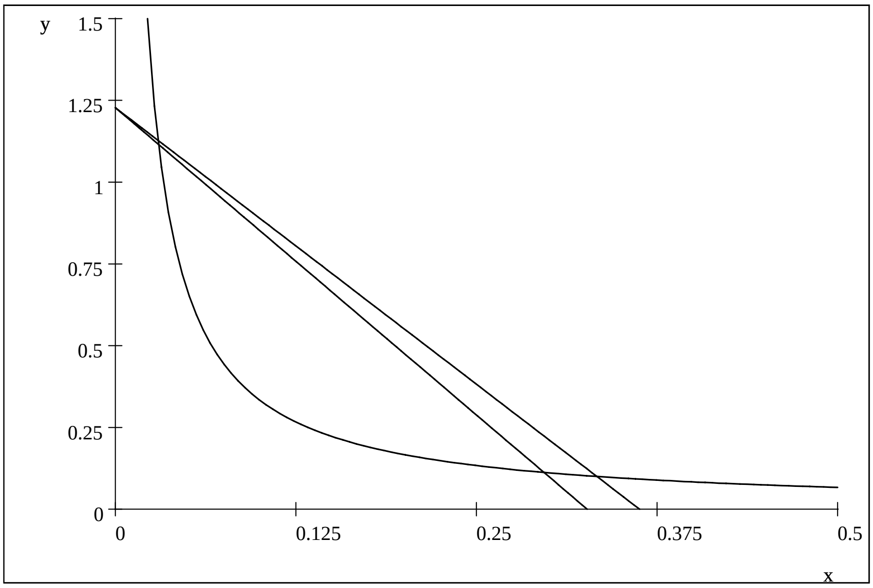


Figure 5b: The effect of a balanced budget expansion ($\gamma = \tau = 0.1$)

An isolated increase in τ shifts the S/y schedule upwards *and* rotate it clockwise. The lower (stable) root falls and the higher (unstable) foot rises. (Figure 5c)

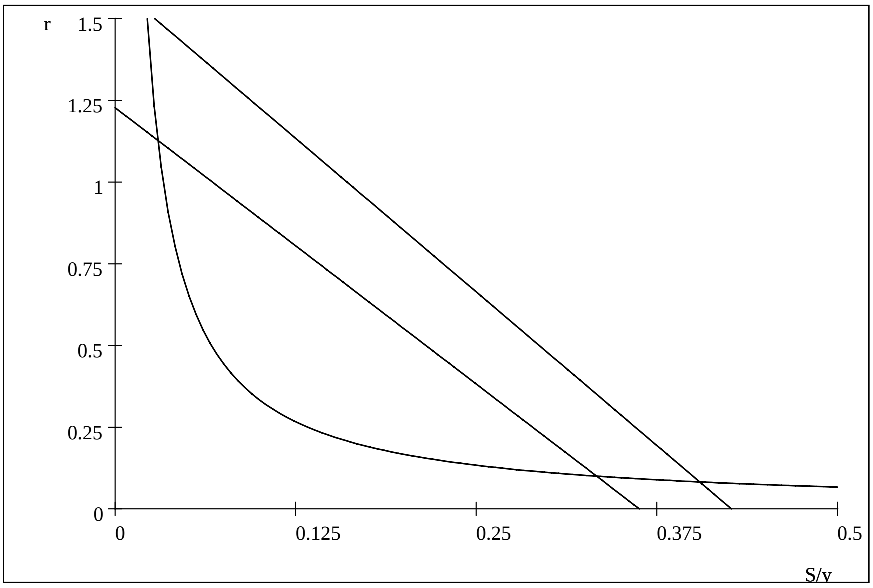


Figure Figure 5c: The effect of an increase in taxation ($\tau = 0.1$)

The results are reproduced below for the following parameter values: $\alpha = 0.333$, $n = 0.10$, $\rho = 0.05$

	$\tau = 0, \gamma = 0$	$\tau = 0.1, \gamma = 0$	$\tau = 0, \gamma = 0.1$	$\tau = 0.1, \gamma = 0.1$
r_1^*	0.10	0.08	0.15	0.11
r_2^*	1.13	1.52	0.73	1.11

Clearly, only the $r^* = n$ solution is consistent with main-stream macroeconomics.

2.7.3 The effect of public debt

Assume that the government has a debt, B, and pursues the policy of keeping the debt constant as a fraction of income, $b = B/y$. This policy implies that the government may increase its debt in each period by $\Delta B_t = B_{t-1} \cdot n$ and only needs collect taxes equal to $T = \tau \cdot y = b \cdot y \cdot (r - n)$, i.e. set the tax rate $\tau = b \cdot (r - n)$,

The I/y schedule is unaffected, but the S/y schedule becomes a non-linear function of r,

$$S/y = f(r) = \left[1 - (1-\alpha) \cdot \left(\frac{1+\rho}{2+\rho} + \frac{1+r}{(2+\rho)\cdot(1+n)}\right)\right] \qquad (18)$$

$$\cdot(1 - b \cdot (r - n)) + b \cdot (r - n) \qquad (19)$$

$$f'(r) < 0 \text{ for } b < \frac{1}{(1+\rho)\cdot(1+n) + (1+r) + r - n} \qquad (20)$$

$$f''(r) = 2 \cdot b \cdot \frac{1-\alpha}{(2+\rho)\cdot(1+n)} \qquad (21)$$

The effect of public debt and a balanced budget policy is depicted in figure 6 for various value of b.For high value of b the model has only one root, $r = n$.

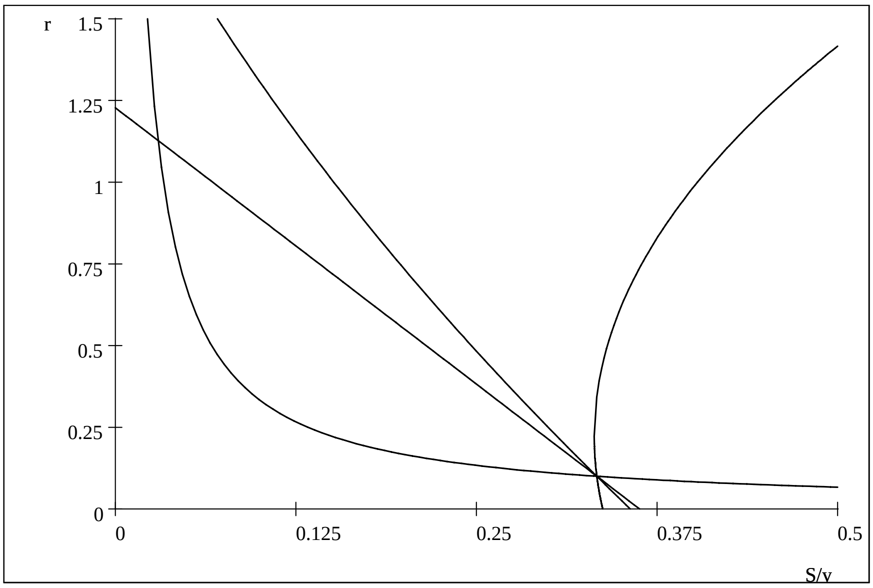


Figure 6: The effect of a constant public debt ($\tau = b \cdot (r - n),\ b = 0;\ b = 0.1;\ b = 0.5)$

References

[1] Diamond, P.A. (1965), National debt in a Neoclassical Growth Model. AER, vol.55.

[2] Lutz, Friedrich A,(1968) The Theory of Interest, Aldine, Chicago

[3] Phelps, E.S. (1961), The Golden Rule of Accunmulation: A Fable for Growthmen. AER, vol.51.

[4] Romer, D. (2001), Advanced Macroeconomics, 2nd. ed., McGraw-Hill

[5] Samuelson, P.A. (1958), An Exact Consumption-Loan Model of Interest With and Without the Social Contrivance of Money. JPE, vol LXVI