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Marko Koethenbueger
Panu Poutvaara

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Marko Koethenbueger

Center for Economic Studies, University of Munich and CESifo

Panu Poutvaara

University of Helsinki, CEBR, CESifo, HECER, IZA and RUESG*

Abstract

We show that taxation of rents may yield an intergenerational Pareto-improvement in a small open economy provided tax revenues are earmarked to reduce wage taxes. Previous literature has shown that rent taxation benefits current young and future generations, while we show that it also benefits the current old generation when the initially prevailing tax mix is sufficiently skewed towards wage taxation.

Keywords: Rent Taxes, Capitalization, Transitional Dynamics, Labor Supply, Asset Prices.

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*Address: Department of Economics, Arkadiankatu 7 (P.O. Box 17), 00014 University of Helsinki, Finland. E-mail: panu.poutvaara@helsinki.fi.

1 Introduction

Rent taxation influences resource allocation through various channels. Feldstein (1977) shows that a rent tax promotes capital accumulation. The rent tax lowers the price of the fixed factor (e.g. land), which reallocates a higher fraction of savings in the households portfolio choice to the accumulation of physical capital. Consequently, welfare of steady state generations rises.¹ The effect is of course non-existent in a small open economy in which the household portfolio choice and domestic capital accumulation are disconnected (e.g. Eaton, 1988). As recently shown by Petrucci (2006), rent taxation may still be beneficial in a small open economy provided households endogenously supply labor. For instance, when rent tax revenues are spent on the reduction of distortionary wage taxes, labor supply increases; an effect which is welcomed by steady state generations. They enjoy a lower wage tax without incurring a drop in the price of their land holdings. The latter cost of rent taxation is borne by transitional generations.

This paper analyzes whether the positive effects of rent taxation extend to transitional generations. We show that, provided the initially prevailing level of wage taxes is sufficiently high, introducing rent taxes to reduce wage taxes increases the sum of rental income and land value of the transitional generation. The rationale is that the rise in labor supply raises the marginal productivity of land which capitalizes in the market price of land. As such, earmarking rent tax revenues is helpful in realizing an intergenerational Pareto-improvement. Rent taxation induces a forward intergenerational transfer from transitional generations to steady state generations. The earmarking simultaneously yields a backward, market-based reaction in asset values, which compensates, possibly to a full extent transitional generations.

¹Among others, Calvo et al. (1979), Chamley and Wright (1987) and Ihori (1990) analyze refinements of the effect.

2 The Model

Consider a small open economy whose population size is normalized at unity. In any period t production combines three input factors: capital, labor and land. The amount of land is normalized to unity. Labor and capital in the economy in period t are denoted by L_t and K_t , respectively. The production function $Y_t = F(L_t, K_t)$ exhibits constant returns to scale in all three factors. Capital is internationally mobile. All markets are competitive, and therefore profit maximization implies that:

$$w_t = F_{L_t}(L_t, K_t), \quad r = F_{K_t}(L_t, K_t). \quad (1)$$

w_t denotes the wage rate in period t and r is the exogenous world interest rate. The land rent in period t , R_t , is given as residual

$$R_t = F(L_t, K_t) - F_{L_t}(L_t, K_t)L_t - F_{K_t}(L_t, K_t)K_t. \quad (2)$$

Individuals can invest their savings in the international capital market or the national land market. The economy produces a composite good, which is a perfect substitute for that produced abroad. Rents are taxed at a rate $\tau^R < 1$. By arbitrage, land value in period t , V_t , is given by

$$(1 + r)V_t = (1 - \tau^R)R_{t+1} + V_{t+1}. \quad (3)$$

Recursive substitution yields:

$$V_t = \sum_{i=1}^{\infty} \frac{(1 - \tau^R)R_{t+i}}{(1 + r)^i}. \quad (4)$$

We analyze an overlapping generations model in which each cohort lives for two periods. Since each cohort consists of homogenous households, we consider a representative household for each cohort. In the first period of their life individuals born in period t choose their labor supply l_t and savings invested in financial assets s_t and land acquisition V_t from the old generation. In the second period of life, individuals receive the rent payment R_{t+1} , sell land to the current young generation and use the receipts along with the deaccumulation

of financial assets $s_t(1+r)$ to finance second-period consumption c_{t+1}^2 . In addition to the rent tax τ^R , the government imposes a tax τ^w on wage income. The first and second period budget constraint thus are

$$(1 - \tau^w)l_t w_t - c_t^1 - s_t - V_t = 0 \quad (5)$$

$$s_t(1+r) + (1 - \tau^R)R_{t+1} + V_{t+1} - c_{t+1}^2 = 0. \quad (6)$$

Household utility is

$$U(1 - l_t, c_t^1, c_{t+1}^2) = c_t^1 + \rho \ln c_{t+1}^2 - \frac{\gamma}{1 + \gamma} l_t^{\frac{1+\gamma}{\gamma}} \quad \rho, \gamma > 0. \quad (7)$$

Households can save and borrow freely at the exogenous interest rate r , determined by the international capital market in order to smoothen their consumption over their lifetime. Labor supply of the young in period t follows from maximizing (7) subject to the budget constraints (5) and (6) which yields

$$l_t = ((1 - \tau^w) w_t)^\gamma.$$

$dl_t/dw_t > 0$ since income effects on labor supply are absent. The elasticity of labor supply with respect to the net-of-tax wage rate is equal to γ .

Land price dynamics are captured by (3). Rearranging terms, all “price-dividend” ratios consistent with arbitrage behavior must satisfy

$$(1+r) \frac{V_t}{R_t} = (1 - \tau^R) + \frac{V_{t+1}}{R_{t+1}}. \quad (8)$$

(8) defines the function $\frac{V_{t+1}}{R_{t+1}} = \phi\left(\frac{V_t}{R_t}\right)$ with $\phi' > 1$. Therefore, a steady state $\frac{V_t}{R_t} = \frac{V_{t+1}}{R_{t+1}}$ exists. Also, the steady state is unique and exhibits point stability. Using (4) the time path of land value is characterized by

$$V_t = \frac{(1 - \tau^R)R_{t+1}}{r}. \quad (9)$$

Any change in land value following a tax reform in period t is captured by a jump in land rents in the subsequent period. Finally, we note that the net foreign assets of the economy in

period t , F_t , satisfy the transversality condition $\lim_{T \rightarrow \infty} \left(\frac{1}{1+r}\right)^T F_{t+T+1} = 0$ as each generation's budget constraint is satisfied over its lifetime and $r > 0$.

3 Rent Tax Reform

We consider a rise in rent taxes at the beginning of period t ; before the young generation supplies labor and the current elderly sell their land to the young generation. The proceeds are used to reduce the wage tax. The current young cohort and the newly born generations benefit from the tax reform. They are subject to a lower wage tax and trade land at the new steady state price. The current old cohort experiences a change in the value of land holdings. To verify whether it is a gain or loss, we first define labor demand, capital demand and the wage rate as a function of the wage tax. The first-order condition for capital demand defines $L_t(K_t)$ and following (2) $R_t(K_t)$. Via the first-order condition for labor demand, we get $w_t(K_t)$. Inserting $L_t(K_t)$ and $w_t(K_t)$ into the labor market clearing condition yields $L_t(K_t) = l_t((1 - \tau^w) w_t(K_t))$ which defines $K_t(\tau^w)$. The slope of the various functions is²

$$\frac{dL_t}{dK_t} = \frac{-F_{KK}}{F_{KL}}, \quad \frac{dR_t}{dK_t} = \frac{L_t \Delta}{F_{KL}}, \quad \frac{dw_t}{dK_t} = \frac{-\Delta}{F_{KL}} \quad \text{and} \quad \frac{dK_t}{d\tau^w} = \frac{w_t(K_t) l'_t}{l'_t(1 - \tau^w) dw_t/dK_t - dL_t/dK_t}, \quad (10)$$

where $\Delta := F_{KK}F_{LL} - F_{KL}^2 > 0$. The public sector budget constraint is $T_t = \tau^w w_t L_t + \tau^R R_t$. Keeping tax revenues constant, tax rates are related as

$$\left. \frac{d\tau^w}{d\tau^R} \right|_{dT_t=0} = - \frac{\partial T_t / \partial \tau^R}{\partial T_t / \partial \tau^w} \quad (11)$$

with

$$\frac{\partial T_t}{\partial \tau^R} = R_t > 0 \quad \text{and} \quad \frac{\partial T_t}{\partial \tau^w} = w_t L_t + \tau^w L_t \frac{dw_t}{d\tau^w} + \tau^w w_t \frac{dL_t}{d\tau^w} + \tau^R \frac{dR_t}{d\tau^w}. \quad (12)$$

We assume that the economy is on the up-ward sloping part of the tax revenue hill, $\partial T_t / \partial \tau^w > 0$. Using (11), (12) and (9) and invoking stationarity of land rents ($R_{t+1} = R_t$) we can compute

² l'_t denotes the derivative of labor supply with respect to the net-of-tax wage rate $(1 - \tau^w) w_t$.

$$\left. \frac{d((1 - \tau^R)R_t + V_t)}{d\tau^R} \right|_{dT_t=0} = -\frac{(1+r)R_t}{r} \left(1 + \frac{(1 - \tau^R) dR_t/d\tau^w}{\partial T_t/\partial \tau^w} \right). \quad (13)$$

The transition generation benefits from the tax reform if and only if (13) is positive. Resorting to a Cobb-Douglas production function with α and β ($\alpha, \beta > 0$, $\alpha + \beta < 1$) being the share of output accruing to labor and capital, we find:

Proposition. *Consider an economy in which $\partial T_t/\partial \tau^w > 0$. There always exists an interval of wage tax rates $(\underline{\tau}^w, \bar{\tau}^w)$, $\underline{\tau}^w < \bar{\tau}^w$ and $\underline{\tau}^w, \bar{\tau}^w \in (0, 1)$, in which changing the tax mix from wage to rent taxation improves welfare of the transition generation.*

Proof. First, assume $\partial T_t/\partial \tau^w > 0$. Inserting (12) into (13) a necessary and sufficient condition for $d((1 - \tau^R)R_t + V_t)/d\tau^R|_{dT_t=0} > 0$ is³

$$w_t L_t + \tau^w L_t \frac{dw_t}{d\tau^w} + \tau^w w_t \frac{dL_t}{d\tau^w} + \frac{dR_t}{d\tau^w} < 0. \quad (14)$$

Using (10) and evaluating the various terms for a Cobb-Douglas production function, (14) holds if and only if

$$\tau^w > \underline{\tau}^w := \frac{1 - \beta - (1 - \alpha - \beta)\omega}{1 - \beta + \omega\alpha}, \quad \omega := \frac{\gamma(1 - \beta)}{1 - \beta + \gamma(1 - \alpha - \beta)}. \quad (15)$$

The assumption $\partial T_t/\partial \tau^w > 0$ holds if and only if

$$\tau^w < \bar{\tau}^w := \frac{1 - \beta - \tau^R(1 - \alpha - \beta)\omega}{1 - \beta + \omega\alpha}. \quad (16)$$

Straightforwardly, $\underline{\tau}^w < \bar{\tau}^w$ as $\tau^R < 1$. We next prove that $\underline{\tau}^w, \bar{\tau}^w \in (0, 1)$. Observe that by (15) and (16) $\frac{di}{d\omega} \frac{d\omega}{d\gamma} < 0$, $i = \underline{\tau}^w, \bar{\tau}^w$. Furthermore, (15) implies $\lim_{\gamma \rightarrow 0} \underline{\tau}^w = 1$ and $\lim_{\gamma \rightarrow \infty} \underline{\tau}^w = 0$. Following (16) $\lim_{\gamma \rightarrow 0} \bar{\tau}^w = 1$ and $\lim_{\gamma \rightarrow \infty} \bar{\tau}^w = \frac{(1 - \tau^R)(1 - \alpha - \beta)}{1 - \beta} \in (0, 1)$. Thus, $\underline{\tau}^w, \bar{\tau}^w \in (0, 1)$ which completes the proof. ■

A rent tax lowers the land value and rental income, ceteris paribus. The budget-balancing reduction in labor taxes, however, increases land productivity in the current and future

³The subsequent proof omits intermediate steps in computing (14), (15) and (16). A detailed proof is in the appendix.

periods. This capitalizes in the land price and may compensate for the negative effect of higher rent taxation, together with the current increase in land rents. In fact, a pre-existing labor tax $\tau^w > \underline{\tau}^w$ generates a sufficiently large distortion in the economy (being convex in the tax rate) so as to render the net effect on land value and rental income positive. The tax reform thereby raises welfare of the transition generation and of steady state generations. The upper bound $\bar{\tau}^w$ ensures that $\partial T_t / \partial \tau^w > 0$. Straightforwardly, for a level of wage taxes above $\bar{\tau}^w$ (and thus $\partial T_t / \partial \tau^w < 0$) it is feasible to lower both the wage and rent tax while leaving tax revenues constant. As a result, current and future generations benefit from the reform. To illustrate the scope for intergenerationally welfare-enhancing policies, consider $(\alpha, \beta, \gamma, \tau^R) = (0.6, 0.3, 0.5, 0.1)$. When evaluated subject to the condition $\partial T_t / \partial \tau^w > 0$ the range of wage tax rates which sustain a Pareto-improvement is $(\underline{\tau}^w, \bar{\tau}^w) = (0.67, 0.71)$. The interval extends to unity in the absence of the condition.

4 Concluding Remarks

The paper shows that rent taxation, when combined with a budget-balancing reduction in wage taxes, may also benefit transitional generations. The commonality of interest between transitional generations and steady state generations becomes weaker when considering alternative fiscal uses of rent tax revenues. For instance, a lump-sum transfer of rent tax receipts to steady state generations is still welcome by them (Petrucci, 2006), but eliminates the market based adjustment in land values in the absence of income effects on labor supply and induces a further downward adjustment of land values when leisure is normal in consumption. We leave a rigorous analysis of alternative uses of rent tax revenues to future research.

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5 Appendix

The appendix contains a detailed proof of the Proposition. For notational simplicity, we omit the time subscript throughout.

Inserting (12) into (13) and invoking stationarity

$$\begin{aligned}
\left. \frac{d((1 - \tau^R)R + V)}{d\tau^R} \right|_{dT=0} &= -\frac{(1+r)R}{r} \left(1 + \frac{(1 - \tau^R) \frac{dR}{d\tau^w}}{wL + \tau^w L \frac{dw}{d\tau^w} + \tau^w w \frac{dL}{d\tau^w} + \tau^R \frac{dR}{d\tau^w}} \right) \\
&= -\frac{(1+r)R}{r} \left(\frac{wL + \tau^w L \frac{dw}{d\tau^w} + \tau^w w \frac{dL}{d\tau^w} + \tau^R \frac{dR}{d\tau^w} + (1 - \tau^R) \frac{dR}{d\tau^w}}{wL + \tau^w L \frac{dw}{d\tau^w} + \tau^w w \frac{dL}{d\tau^w} + \tau^R \frac{dR}{d\tau^w}} \right) \\
&= -\frac{(1+r)R}{r} \left(\frac{wL + \tau^w L \frac{dw}{d\tau^w} + \tau^w w \frac{dL}{d\tau^w} + \frac{dR}{d\tau^w}}{wL + \tau^w L \frac{dw}{d\tau^w} + \tau^w w \frac{dL}{d\tau^w} + \tau^R \frac{dR}{d\tau^w}} \right). \tag{17}
\end{aligned}$$

Assuming $\partial T / \partial \tau^w = wL + \tau^w L \frac{dw}{d\tau^w} + \tau^w w \frac{dL}{d\tau^w} + \tau^R \frac{dR}{d\tau^w} > 0$, a necessary and sufficient condition for (17) to be positive is

$$wL + \tau^w L \frac{dw}{d\tau^w} + \tau^w w \frac{dL}{d\tau^w} + \frac{dR}{d\tau^w} < 0.$$

Using the chain rule the condition reads

$$wL + \left(\tau^w L \frac{dw}{dK} + \tau^w w \frac{dL}{dK} + \frac{dR}{dK} \right) \frac{dK}{d\tau^w} < 0. \tag{18}$$

Evaluating the responses $\frac{di}{dK}$, $i = w, L, K$ (see (10)) for the Cobb-Douglas production function $Y = L^\alpha K^\beta$

$$(18) = \alpha L^\alpha K^\beta + \left(\tau^w w \frac{1-\beta}{\alpha} \frac{L}{K} + (1-\tau^w)(1-\alpha-\beta) L^\alpha K^{\beta-1} \right) \frac{dK}{d\tau^w} \quad (19)$$

Inserting $w = \alpha L^{\alpha-1} K^\beta$ and collecting terms

$$(19) = \alpha L^\alpha K^\beta + (\tau^w(1-\beta) + (1-\tau^w)(1-\alpha-\beta)) L^\alpha K^{\beta-1} \frac{dK}{d\tau^w}. \quad (20)$$

Using the first-order condition for capital demand, $r = \beta L^\alpha K^{\beta-1}$, to substitute for K , and rearranging yields

$$(20) = \alpha \left(\frac{\beta}{r} \right)^{\frac{\beta}{1-\beta}} L^{\frac{\alpha}{1-\beta}} + (1-\alpha-\beta + \tau^w \alpha) \left(\frac{\beta}{r} \right)^{-1} \frac{dK}{d\tau^w}. \quad (21)$$

We decompose $\frac{dK}{d\tau^w}$ into $\frac{dK}{dL} \frac{dL}{d\tau^w}$. By the first-order condition $r = \beta L^\alpha K^{\beta-1}$ we have

$$\frac{dK}{dL} = \frac{\alpha}{1-\beta} \left(\frac{\beta}{r} \right)^{\frac{1}{1-\beta}} L^{\frac{\alpha}{1-\beta}-1}. \quad (22)$$

Furthermore, labor supply is $l = ((1-\tau^w)w_t)^\gamma$. Substituting w by the first-order condition $w = \alpha L^{\alpha-1} K^\beta$ and, subsequently, K by the (inverted) first-order condition $r = \beta L^\alpha K^{\beta-1}$,

$$l = \left((1-\tau^w) \alpha \left(\frac{\beta}{r} \right)^{\frac{\beta}{1-\beta}} L^{\frac{\alpha}{1-\beta}-1} \right)^\gamma.$$

Setting $l = L$ and solving for L yields

$$L = \left((1-\tau^w) \alpha \left(\frac{\beta}{r} \right)^{\frac{\beta}{1-\beta}} \right)^\omega, \quad \omega := \frac{\gamma(1-\beta)}{1-\beta + \gamma(1-\alpha-\beta)}.$$

Taking the derivative

$$\begin{aligned} \frac{dL}{d\tau^w} &= -\omega \left((1-\tau^w) \alpha \left(\frac{\beta}{r} \right)^{\frac{\beta}{1-\beta}} \right)^{\omega-1} \alpha \left(\frac{\beta}{r} \right)^{\frac{\beta}{1-\beta}} \\ &= -\omega \frac{1}{1-\tau^w} L. \end{aligned} \quad (23)$$

Inserting (22) and (23) into (21) we get

$$\begin{aligned}
(21) &= \alpha \left(\frac{\beta}{r}\right)^{\frac{\beta}{1-\beta}} L^{\frac{\alpha}{1-\beta}} - (1 - \alpha - \beta + \tau^w \alpha) \left(\frac{\beta}{r}\right)^{-1} \frac{\alpha}{1-\beta} \left(\frac{\beta}{r}\right)^{\frac{1}{1-\beta}} L^{\frac{\alpha}{1-\beta}-1} \omega \frac{1}{1-\tau^w} L \\
&= \alpha \left(\frac{\beta}{r}\right)^{\frac{\beta}{1-\beta}} L^{\frac{\alpha}{1-\beta}} - (1 - \alpha - \beta + \tau^w \alpha) \frac{\alpha}{1-\beta} \left(\frac{\beta}{r}\right)^{\frac{\beta}{1-\beta}} L^{\frac{\alpha}{1-\beta}} \omega \frac{1}{1-\tau^w} \\
&= \alpha \left(\frac{\beta}{r}\right)^{\frac{\beta}{1-\beta}} L^{\frac{\alpha}{1-\beta}} \left(1 - (1 - \alpha - \beta + \tau^w \alpha) \frac{1}{1-\beta} \omega \frac{1}{1-\tau^w}\right).
\end{aligned}$$

Recall, provided $\partial T/\partial \tau^w > 0$ the sum of rental income and land value of the transitional generation increases, $d((1 - \tau^R)R_t + V_t)/d\tau^R|_{dT_t=0} > 0$, if and only if

$$1 - (1 - \alpha - \beta + \tau^w \alpha) \frac{1}{1-\beta} \omega \frac{1}{1-\tau^w} < 0. \quad (24)$$

Equivalently stated,

$$\tau^w > \underline{\tau}^w := \frac{1 - \beta - (1 - \alpha - \beta)\omega}{1 - \beta + \omega\alpha}, \quad \omega := \frac{\gamma(1 - \beta)}{1 - \beta + \gamma(1 - \alpha - \beta)}. \quad (25)$$

We next derive the condition under which

$$\frac{\partial T}{\partial \tau^w} = wL + \tau^w L \frac{dw}{d\tau^w} + \tau^w w \frac{dL}{d\tau^w} + \tau^R \frac{dR}{d\tau^w} > 0$$

holds. As can be inferred from (17) the expression is almost congruent to the term $wL + \tau^w L \frac{dw}{d\tau^w} + \tau^w w \frac{dL}{d\tau^w} + \frac{dR}{d\tau^w}$ which we stepwise rearranged to arrive at (25). Reiterating the same steps, the condition for $\partial T_t/\partial \tau^w > 0$ reads

$$\tau^w < \bar{\tau}^w := \frac{1 - \beta - \tau^R(1 - \alpha - \beta)\omega}{1 - \beta + \omega\alpha}, \quad \omega := \frac{\gamma(1 - \beta)}{1 - \beta + \gamma(1 - \alpha - \beta)}.$$

Straightforwardly, $\underline{\tau}^w < \bar{\tau}^w$ when $\tau^R < 1$. A change in the tax mix from wage to rent taxation increases land value if and only if $\tau^w \in (\underline{\tau}^w, \bar{\tau}^w)$.

Helpful in proving that $\underline{\tau}^w, \bar{\tau}^w \in (0, 1)$ we first compute the derivative

$$\begin{aligned}
\frac{d\omega}{d\gamma} &= \frac{(1 - \beta)(1 - \beta + \gamma(1 - \alpha - \beta)) - \gamma(1 - \alpha - \beta)}{(1 - \beta + \gamma(1 - \alpha - \beta))^2} \\
&= \frac{(1 - \beta)^2}{(1 - \beta + \gamma(1 - \alpha - \beta))^2} > 0.
\end{aligned}$$

Turning to the slope of $\underline{\tau}^w$ the with respect to ω

$$\begin{aligned}\frac{d\underline{\tau}^w}{d\omega} &= \frac{-(1-\alpha-\beta)(1-\beta+\omega\alpha) - (1-\beta - (1-\alpha-\beta)\omega)\alpha}{(1-(1-\omega)\alpha)^2} \\ &= \frac{-(1-\beta)^2}{(1-(1-\omega)\alpha)^2} < 0.\end{aligned}$$

Similarly,

$$\begin{aligned}\frac{d\bar{\tau}^w}{d\omega} &= \frac{-\tau^R(1-\alpha-\beta)(1-\beta+\omega\alpha) - (1-\beta - \tau^R(1-\alpha-\beta)\omega)\alpha}{(1-(1-\omega)\alpha)^2} \\ &= \frac{-(1-\beta)(\alpha + \tau^R(1-\alpha-\beta))}{(1-(1-\omega)\alpha)^2} < 0.\end{aligned}$$

Therefore,

$$\frac{di}{d\omega} \frac{d\omega}{d\gamma} < 0, \quad i = \underline{\tau}^w, \bar{\tau}^w.$$

To determine the maximal and minimal value of $\underline{\tau}^w$ and $\bar{\tau}^w$, we first observe that

$$\lim_{\gamma \rightarrow 0} \omega = 0 \tag{26}$$

and applying L'Hôpital's rule

$$\lim_{\gamma \rightarrow \infty} \omega = \frac{1-\beta}{1-\alpha-\beta}. \tag{27}$$

Given by (26) and (27)

$$\lim_{\gamma \rightarrow 0} \underline{\tau}^w = 1 \quad \text{and} \quad \lim_{\gamma \rightarrow \infty} \underline{\tau}^w = 0$$

and

$$\lim_{\gamma \rightarrow 0} \bar{\tau}^w = 1 \quad \text{and} \quad \lim_{\gamma \rightarrow \infty} \bar{\tau}^w = \frac{(1-\tau^R)(1-\alpha-\beta)}{1-\beta} \in (0, 1).$$

Thus, $\underline{\tau}^w, \bar{\tau}^w \in (0, 1)$ which completes the proof.